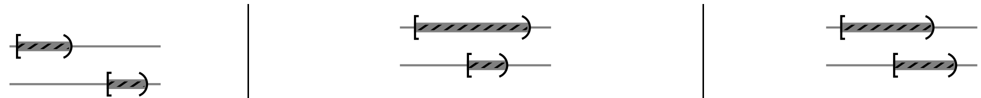
1.

(a) Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a subset. The criterion that $\mathcal{B}$ be a base for the topology it generates is that the intersection of any two members of $\mathcal{B}$ be a union of elements of $\mathcal{B}$. In this case, the set $\mathcal{B}$ satisfies a stronger property : the intersection of two elements of $\mathcal{B}$ is (if nonempty) an element of $\mathcal{B}$ – no further union necessary.

There are three ways that two subsets of the form $[a_1, b_1)$, $[a_2, b_2)$ can intersect :



Disjoint intervals | Complete containment of one in the other | Overlapping intervals

In the first case, the intersection is empty, but in the second two the intersection is again of the form $[a_3, b_3)$, and so an element of $\mathcal{B}$.

(b) Given (a, b) with $a < b$, choose $n_0 > 0$ large enough so that $a + \frac{1}{n_0} < b$. Then

$$(a, b) = \bigcup_{n \geqslant n_0} [a + \tfrac{1}{n}, b),$$

and so (a, b) is an open set in the topology τ_S.

(c) The complement of $[a, b)$ is $(-\infty, a) \cup [b, \infty)$. Since

$$(-\infty, a) = \bigcup_{n \in \mathbb{N}, n>0} [a - n, a) \quad \text{and} \quad [b, \infty) = \bigcup_{n \in \mathbb{N}, n>0} [b, a + n),$$

the sets $(-\infty, a)$ and $[b, \infty)$ are open in τ_S, since each is a union of sets in $\mathcal{B}$.

Therefore $(-\infty, a) \cup [b, \infty)$ is open in τ_S, and so $[a, b)$ closed.

(d) The complement of $\{x_0\}$ is $(-\infty, x_0) \cup (x_0, \infty)$. The set $(-\infty, x_0)$ is open by the argument from (c), and similarly since $[x_0, \infty)$ is open by (c), the set

$$(x_0, \infty) = \bigcup_{n \in \mathbb{N}, n>0} [x_0 + \frac{1}{n}, \infty)$$

is open in τ_S. Therefore $\{x_0\}$ is a closed set.

(e) However, the set $\{x_0\}$ is not open. $\mathcal{B}$ is a base for τ_S, and so a set U is open in τ_S if and only if U is a union of sets in $\mathcal{B}$. However no set in $\mathcal{B}$ is a subset of $\{x_0\}$ (for instance, every set in $\mathcal{B}$ is infinite, and $\{x_0\}$ is finite), and so $\{x_0\}$ cannot be written as a union of elements of $\mathcal{B}$. I.e., $\{x_0\}$ is not an open subset in τ_S.

(f) In the discrete topology, every subset is both open and closed. Since $\{x_0\}$ is not open, τ_S is not the discrete topology.

(g) By (d) and (e), $\complement_{\mathbb{R}}\{x_0\}$ is a set which is open in τ_S, but not closed. So, not every open set in τ_S is closed.

(h) Fix $x_0 \in \mathbb{R}$. Then $\{x_0\} = [x_0 - 1, x_0] \cap [x_0, x_0 + 1]$, and so $\{x_0\}$ is an open set in the topology generated by $\mathcal{B}''$. Since $x_0 \in \mathbb{R}$ was arbitrary, this is true for each $x_0 \in \mathbb{R}$. But every set $A \subseteq \mathbb{R}$ is the union

$$S = \bigcup_{x \in S} \{x\},$$

so every set $S \subseteq \mathbb{R}$ is open. Thus the topology generated by $\mathcal{B}''$ is the discrete topology on $\mathbb{R}$.

2.

(a) Since $c(\varnothing) = \varnothing$ (by (K1)), $\varnothing$ is a closed set.

(b) By (K2) we have $X \subseteq c(X)$, and since output of c is a subset of X, we have $c(X) \subseteq X$. Therefore $c(X) = X$, and X is a closed set.

(c) Let Z_1 and Z_2 be closed sets. Then by (K4) we have

$$c(Z_1 \cup Z_2) = c(Z_1) \cup c(Z_2) = Z_1 \cup Z_2,$$

where in the last equality we have used the fact that Z_1 and Z_2 are closed. Thus, the union of two closed sets is always closed. By induction it follows that the union of finitely many closed sets is closed.

(d) Writing $B = A \cup D$, by (K4) we have

$$c(B) = c(A) \cup c(D),$$

and thus $c(A) \subseteq c(B)$.

(e) Since W is the intersection of the Z_i, $W \subseteq Z_i$ for each $i \in I$.

(f) Using (d) we therefore have $c(W) \subseteq c(Z_i) = Z_i$ for each $i \in I$, where in the last equality we have used the fact that Z_i is closed.

(g) Since $c(W) \subseteq Z_i$ for each i, we have $c(W) \subseteq \bigcap_{i \in i} Z_i = W$.

(h) by (K2), $W \subseteq c(W)$, and so combined with (g) we get $c(W) = W$.

(i) Given any $A \subseteq X$, and using the topology determined by the closure operator, we have that $c(\overline{A}) = \overline{A}$ (since $\overline{A}$ is a closed set in this topology) and $A \subseteq \overline{A}$ (property (cl2)). Applying c to this last inclusion, and using (K2) we get $c(A) \subseteq c(\overline{A}) = \overline{A}$.

On the other hand, by (K3) $c(c(A)) = c(A)$, i.e., $c(A)$ is a closed set, which by (K2) contains A. Since $\overline{A}$ is contained in every closed set which contains A, we have $\overline{A} \subseteq c(A)$. Combined with the previous inclusion this gives $c(A) = \overline{A}$.

NOTE : (K1) is only needed to show (a), and (K3) is only needed to show (i).