1.

(a) Suppose that $f\colon X \longrightarrow Y$ is injective, and g_1, $g_2\colon Z \longrightarrow X$ two maps so that $f \circ g_1 = f \circ g_2$.

For each $z \in Z$, this means that $f(g_1(z)) = f(g_2(z))$. Since f is injective, $f(g_1(z)) = f(g_2(z))$ can only happen when $g_1(z) = g_2(z)$. Since, for each $z \in Z$, $g_1(z) = g_2(z)$, $g_1 = g_2$.

(b) Suppose that $f\colon X \longrightarrow Y$ is not injective. Then there are two elements x_1, $x_2 \in X$, with $x_1 \neq x_2$, so that $f(x_1) = f(x_2)$.

Let $Z = \{a\}$, i.e., let Z be a set with the single element a. Define the function $g_1\colon Z \longrightarrow X$ by $g_1(a) = x_1$, and define $g_2\colon Z \longrightarrow X$ by $g_2(a) = x_2$.

Then, $f(g_1(a)) = f(x_1) = f(x_2) = f(g_2(a))$. Since a is the only element of Z, this means that $f \circ g_1 = f \circ g_2$. But $g_1 \neq g_2$, since $g_1(a) \neq g_2(a)$. Therefore f is not a monomorphism in **Set**.

(c) Suppose that $f\colon X \longrightarrow Y$ is a surjective map, and g_1, $g_2\colon Y \longrightarrow Z$ two maps so that $g_1 \circ f = g_2 \circ f$.

Pick $y \in Y$. Since f is surjective, there is an $x \in X$ so that $f(x) = y$. Then

$$g_1(y) = g_1(f(x)) = (g_1 \circ f)(x) = (g_2 \circ f)(x) = g_2(f(x)) = g_2(y).$$

Since $y \in Y$ was arbitrary, this is true for all $y \in Y$, and so $g_1 = g_2$.

(d) Suppose that $f\colon X \longrightarrow Y$ is a map of sets which is *not* surjective. Let $A = f(x)$ (i.e, $A = \operatorname{Im}(f)$) and set $B = \complement_Y A$. Since f is not surjective, A is not all of Y, and B is nonempty.

Let $Z = \{z_1, z_2\}$, i.e., let Z be a set with two elements. Define a map $g_1\colon Y \longrightarrow Z$ by $g_1(y) = z_1$ for all $y \in Y$, and define $g_2\colon Y \longrightarrow Z$ by

$$g_2(y) = \begin{cases} z_1 & \text{if } y \in A \\ z_2 & \text{if } y \notin A \end{cases}$$

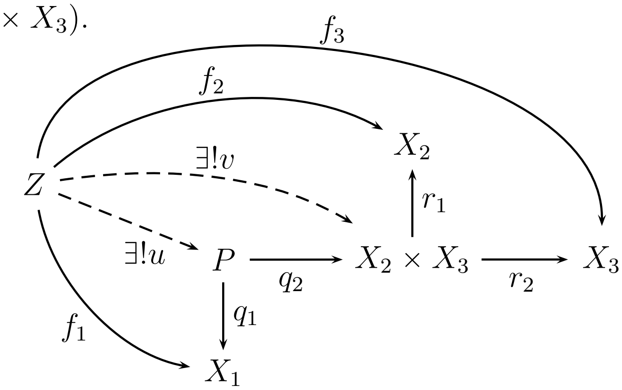
Since B is not empty, there is at least one $y \in B$, and for that y, $g_1(y) = z_1$, $g_2(y) = z_2$, and so $g_1 \neq g_2$.

But, for all $x \in X$, $g_1(f(x)) = z_1 = g_2(f(x))$, since for all $x \in X$, $f(x) \in A$. Thus, $g_1 \circ f = g_2 \circ f$, but $g_1 \neq g_2$, and so f is not an epimorphism in **Set**.

2.

(a) To slightly reduce space, set $P = X_1 \times (X_2 \times X_3)$.

Recall that $p_1 = q_1$, $p_2 = r_1 \circ q_2$, and $p_3 = r_2 \circ q_2$.



Let Z be a set, and $f_i \colon Z \longrightarrow X_i$, $i \in \{1, 2, 3\}$ be maps of sets. By the universal property of $X_2 \times X_3$ there exists a unique map $v \colon Z \longrightarrow X_2 \times X_3$ such that $r_1 \circ v = f_2$ and $r_2 \circ v = f_3$. By the universal property of P there exists a unique map $u \colon Z \longrightarrow P$ such that $q_1 \circ u = f_1$ and $q_2 \circ u = v$.

But now

$$\begin{aligned} p_1 \circ u &= q_1 \circ u = f_1, \\ p_2 \circ u &= r_1 \circ q_2 \circ u = r_1 \circ v = f_2, \quad \text{and} \\ p_3 \circ u &= r_2 \circ q_2 \circ u = r_2 \circ v = f_3. \end{aligned}$$

This shows that given f_1, f_2, and f_3, there is a map $u \colon Z \longrightarrow P$ such that $p_i \circ u = f_i$ for $i \in \{1, 2, 3\}$.

To see that this map is unique, suppose that there are two such maps, u_1 and u_2. The map $q_2 \circ u_1$ is a map $Z \longrightarrow X_2 \times X_3$ with the property that, composed with r_1 and r_2, we get f_2 and f_3 respectively. Similarly $q_2 \circ u_2$ also has this property. By the uniqueness of maps to $X_2 \times X_2$, we must have $q_2 \circ u_1 = q_2 \circ u_2$. But we also have $q_1 \circ u_1 = f_1 = q_1 \circ u_2$. That is, u_1 and u_2 composed with q_1 and q_2 respectively are also the same maps. By the uniqueness of maps to $X_1 \times (X_2 \times X_3)$, we must have $u_1 = u_2$.

Hence P satisfies the universal property of the triple product. □

(b) The map $f \colon X_1 \times X_2 \times X_3 \longrightarrow X_1 \times (X_2 \times X_3)$ is $f(x_1, x_2, x_3) = (x_1, (x_2, x_3))$.

These triple products are isomorphic, via a canonical isomorphism, but not equal.