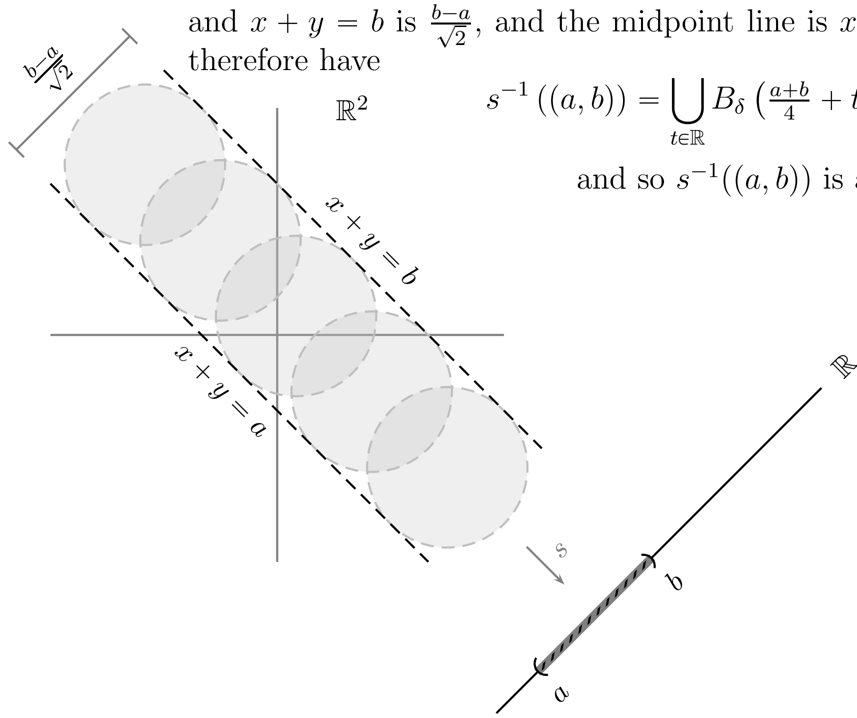
1.

(a) It is sufficient to check continuity using a base for the topology on $\mathbb{R}$. Let $(a, b) \subseteq \mathbb{R}$ be an open interval. In $\mathbb{R}^2$, the Euclidean distance between the lines $x + y = a$ and $x + y = b$ is $\frac{b-a}{\sqrt{2}}$, and the midpoint line is $x + y = \frac{a+b}{2}$. Setting $\delta = \frac{b-a}{2\sqrt{2}}$ we therefore have

$$s^{-1}((a, b)) = \bigcup_{t \in \mathbb{R}} B_\delta\left(\tfrac{a+b}{4} + t, \tfrac{a+b}{4} - t\right),$$

and so $s^{-1}((a, b))$ is an open set in $\mathbb{R}^2$.



(b)

$$\begin{aligned} |(x_0 + \delta_x)(y_0 + \delta_y) - x_0 y_0)| &= |x_0\delta_y + y_0\delta_x + \delta_x\delta_y| \leqslant |x_0||\delta_y| + |y_0||\delta_x| + |\delta_x||\delta_y| \\ &\leqslant \frac{|x_0|\epsilon}{3(1 + |x_0|)} + \frac{|y_0|\epsilon}{3(1 + |y_0|)} + \frac{\epsilon^2}{9(1 + |x_0|)(1 + |y_0|)} \\ &\leqslant \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

In the last line we have used the fact that $\epsilon \leqslant 3$ to deduce that $\frac{\epsilon \cdot \epsilon}{9} \leqslant \frac{\epsilon}{3}$.

(c) Let $V \subseteq \mathbb{R}$ be an open subset, $(x_0, y_0) \in p^{-1}(V)$, and set $w_0 = p(x_0, y_0) \in V$. Since V is open, there is an $\epsilon > 0$ so that $(w_0 - \epsilon, w_0 + \epsilon) \in V$. We may assume that $\epsilon \leqslant 3$, since shrinking the interval will not change the fact that it is contained in V. Let U be the open box

$$U = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, |x - x_0| < \frac{\epsilon}{3(1 + |y_0|)},\ |y - y_0| < \frac{\epsilon}{3(1 + |x_0|)} \right\}.$$

Pick $(x, y) \in U$, and set $\delta_x = x - x_0$ and $\delta_y = y - y_0$. By the definition of U we have $|\delta_x| \leqslant \frac{\epsilon}{3(1+|y_0|)}$ and $|\delta_y| \leqslant \frac{\epsilon}{3(1+|x_0|)}$. By (b) we therefore have

$$|p(x, y) - w_0| = |(x_0 + \delta_x)(y_0 + \delta_y) - x_0 y_0| \leqslant \epsilon,$$

and thus $p(x, y) \in (w_0 - \epsilon, w_0 + \epsilon) \subseteq V$. Since $(x, y) \in U$ was arbitrary, $p(U) \subseteq V$, or $U \subseteq p^{-1}(V)$.

This shows that (x_0, y_0) is contained in an open set contained in $p^{-1}(V)$. Since $(x_0, y_0) \in p^{-1}(V)$ was arbitrary, we conclude that $p^{-1}(V)$ is open. Thus p is a continuous function.

(d) Since u, s, and p are continuous, $s \circ u$ and $p \circ u$ are continuous. But

$$(s \circ u)(x) = s(f(x), g(x)) = f(x) + g(x) = (f + g)(x)$$

and

$$(p \circ u)(x) = p(f(x), g(x)) = f(x) \cdot g(x) = (f \cdot g)(x).$$

I.e., $s \circ u$ and $p \circ u$ are the sum and product of f and g. By the composition argument above, the sum and product are continuous.

(e) Part (d) shows that the sum and product of continuous functions are again continuous functions. The constant function 1 is also continuous (every constant function is continuous). It is clear that the sum and product satisfy the axioms for a ring (e.g., distributivity of multiplication over addition) since the sum and product operations are defined pointwise, and those axioms hold in $\mathbb{R}$. Thus $\mathcal{C}(X, \mathbb{R})$ is a ring.

2.

(a) The function f_n is "transate by $-\frac{1}{2n}$. This function is a bijection with inverse function "translate by $\frac{1}{2n}$. Therefore $f_n^{-1}((0,1))$ is the interval obtained by translating $(0, 1)$ by $\frac{1}{2n}$, i.e., the interval $\left(\frac{1}{2n},\ 1 + \frac{1}{2n}\right)$.

(b) The function u is given by $u(x) = (f_1(x), f_2(x), f_3(x),\ f_4(x), \ldots)$. The set U is the set where every coordinate is in $(0, 1)$. Therefore $x \in u^{-1}(U)$ if and only if $f_i(x) \in (0, 1)$ for each n, and so $u^{-1}(U) = \bigcap_{n \in I} f_n^{-1}((0, 1))$.

Here is the same argument, expressed another way. The set U is the intersection $\bigcap_{n \geqslant 1} p_n^{-1}((0, 1))$, where p_n is the n-th projection map. Since $p_n \circ u = f_n$, we get that

$$u^{-1}(U) = u^{-1}\left(\bigcap_{n \geqslant 1} p_n^{-1}((0,1)\right) = \bigcap_{n \geqslant 1} u^{-1}(p_n^{-1}((0,1)) = \bigcap_{n \geqslant 1} (p_n \circ u)^{-1}((0,1)) = \bigcap_{n \geqslant 1} f_n^{-1}((0,1)).$$

(c) The intersection in (b) is the set $(\frac{1}{2}, 1]$.

To see this we note that the interval $(\frac{1}{2}, 1]$ is contained in each interval $\left(\frac{1}{2n},\ 1 + \frac{1}{2n}\right)$, that $\frac{1}{2}$ or any number less than $\frac{1}{2}$, is not contained in $f_1^{-1}((0,1)) = (\frac{1}{2}, \frac{3}{2})$, and that for any number $z > 1$, picking n large enough so that $1 + \frac{1}{2n} < z$, z is not in $f_n^{-1}((0,1))$. I.e., the interval $(\frac{1}{2}, 1]$ is contained in each $f_n^{-1}((0,1))$, and nothing outside $(\frac{1}{2}, 1]$ is contained in all of the $f_n^{-1}((0,1))$.

(d) The set $(\frac{1}{2}, 1]$ is not open in the standard topology on $\mathbb{R}$.

(e) Suppose that we choose a topology on $\mathbb{R}^\infty$ which has the set U as an open subset. Each of the functions $f_n \colon \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. So, if $\mathbb{R}^\infty$ is the product in the category of topological spaces, the map $u \colon \mathbb{R} \longrightarrow \mathbb{R}^\infty$ given by $u(x) = (f_1(x), f_2(x), f_3(x), \ldots)$ has to be continuous. (This is the only map u so that $p_n \circ u = f_n$ for each n.) Then $u^{-1}(U)$ would have to be an open set in $\mathbb{R}$, and we have just see that it is not.

Therefore, $\mathbb{R}^\infty$ with any topology containing U as an open set is not the product in the category of topological spaces.

(f) By the same reasoning as in (b), $u^{-1}(V)$ is the intersection of all the $f_n^{-1}(B_n)$, with $B_n = (0,1)$ if $n \in \{1, 2, 4\}$ and $B_n = \mathbb{R}$ otherwise. Since $f_n^{-1}(\mathbb{R}) = \mathbb{R}$, intersecting with these open sets doesn't change anything. Therefore

$$\begin{aligned} u^{-1}(V) &= \bigcap_{n \geqslant 1} f_n^{-1}(B_n) = f_1^{-1}((0,1)) \cap f_2^{-1}((0,1)) \cap f_4^{-1}((0,1)) \\ &= \left(\tfrac{1}{2}, \tfrac{3}{2}\right) \cap \left(\tfrac{1}{4}, \tfrac{5}{4}\right) \cap \left(\tfrac{1}{8}, \tfrac{9}{8}\right) = \left(\tfrac{1}{2},\ \tfrac{9}{8}\right). \end{aligned}$$

This set is an open set in $\mathbb{R}$.

NOTE : This problem helps explain why, when there are infinitely many factors, the box topology is not the right one to make the product set into the product in the category of topological spaces.

Suppose that X_i, $i \in I$ are topological spaces, and set $P = \prod_{i \in I} X_i$. Given another topological space Z, and continuous maps $f_i \colon Z \longrightarrow X_i$ for each i, we get a map $u \colon Z \longrightarrow P$, given by $u(z) = (f_i(z))_{i \in I}$, i.e., the map u such that $p_i \circ u = f_i$ for each $i \in I$. If we want P to have the universal property of the product, then we want u to be continuous.

Suppose we have a product set $U = \prod_{i \in I} B_i \subseteq P$. Then $u^{-1}(U) = \bigcap_{i \in I} f_i^{-1}(B_i)$. If each $B_i \subseteq X_i$ is open, then each $f_i^{-1}(B_i)$ is an open set in Z. If we want U to be in the topology on P, and want u to be continuous, this infinite intersection of open sets must still be open.

But, having an infinite intersection of open sets be open is not part of the axioms of a topology, and the only way to guarantee that this infinite intersection is open is to make sure it is reall always a finite intersection. We can do that by insisting that there are only finitely many i so that $B_i \neq X_i$. When $B_i = X_i$, $f_i^{-1}(B_i) = Z$, and intersecting a subset of Z with Z does not change anything.

That is, if we let $J \subseteq I$ be the (finite) set of indices i so that $B_i \neq X_i$, then

$$\bigcap_{i \in I} f_i^{-1}(B_i) = \bigcap_{i \in J} f_i^{-1}(B_i),$$

and, since J is a finite set, the latter is a finite intersection of open sets, which is therefore open by the axioms of a topology.

3.

(a) Since X has the discrete topology, each of the sets $\{0\}$ and $\{1\}$ are open in X. Therefore, given $\underline{x} = (x_0, x_1, x_2, \ldots)$, setting $B_i = \{x_i\}$, we have that each B_i is open in X, and so

$$\{\underline{x}\} = \prod_{i \in \mathbb{N}} \{x_i\} \in \mathcal{B}'.$$

Therefore $\{\underline{x}\}$ is an open subset of P in the box topology. Since this is true for each $\underline{x} \in P$, the box topology on P is, in this case, the discrete topology.

(b) Let $U \subseteq P$ be a nonempty open set in the product topology, and $\underline{x} \in U$. Since $\mathcal{B}$ is a base for the product topology, U is a union of elements of $\mathcal{B}$. In particular, there is some $V \in \mathcal{B}$ such that $V \subseteq U$ and $\underline{x} \in V$. By the description of $\mathcal{B}$,

$$V = \prod_{i \in \mathbb{N}} B_i$$

with $B_i = X$ for all but finitely many i. Let $J \subseteq \mathbb{N}$ be the finite set where $B_j \neq X$.

Since the only nonempty open sets in X are X, $\{0\}$ and $\{1\}$, for each $j \in J$ we have that B_j is either $\{0\}$ or $\{1\}$. Since $\underline{x} \in V$, for each $j \in J$ it must be that $B_j = \{x_j\}$.

A point $\underline{y} = (y_0, y_1, \ldots)$ is in V if and only if $y_i \in B_i$ for each i. For $i \notin J$ this is no condition : $B_i = X$. For $j \in J$, the condition is that $y_j \in B_j = \{x_j\}$, so $y_j = x_j$. I.e., the set $V = \prod_{i \in \mathbb{N}} B_i$ can also be described as

$$V = \left\{ \underline{y} = (y_0, y_1, y_2, \ldots) \in P \;\middle|\; y_j = x_j \text{ for all } j \in J \right\}$$

(for the set J we chose above).

Since we know that V is an open set (V is part of the base for the product topology), and contained in U, this answers question (b).

(c) From the above description, it is clear that V is infinite, and since $V \subseteq U$, U is infinite too. Thus, in the product topology, all nonempty open sets are infinite. Since in the box topology (in this example) every set, including finite sets, are open, the box topology and the product topology are different.