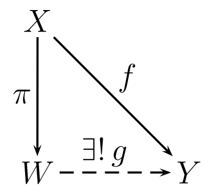
1.

(a) A pair (W, π) is a quotient of a topological space X by a relation R if it satisfies the following two conditions.



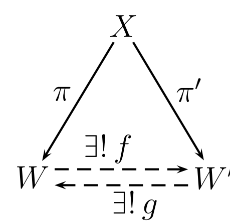
(a1) For each $x_1, x_2 \in X$, if $x_1 \sim_R x_2$ then $\pi(x_1) = \pi(x_2)$.

(a2) (W, π) is *universal* with respect to the property (a1) : If $f \colon X \longrightarrow Y$ is a morphism such that $x_1 \sim_R x_2 \implies f(x_1) = f(x_2)$, then there is a unique morphism $g \colon W \longrightarrow Y$ with the property that $f = g \circ \pi$.

(Note that since we are working in the category of topological spaces, "morphism" means "continuous map".)

(b) Let us say that a morphism $p \colon X \longrightarrow Y$ has property $(*)$ if

$(*)$ for all $x_1, x_2 \in X$, $x_1 \sim_R x_2 \implies p(x_1) = p(x_2)$.



Given two quotients (W, π) and (W', π'), since $\pi' \colon X \longrightarrow W'$ has property $(*)$, by the universal property of (W, π) there exists a unique morphism $f \colon W \longrightarrow W'$ so that $\pi' = f \circ \pi$.

Similarly, since $\pi \colon X \longrightarrow W$ has property $(*)$, by the universal property of W' there exists a unique morphism $g \colon W' \longrightarrow W$ so that $\pi = g \circ \pi'$.

Let us now consider the composition $(g \circ f) \colon W \longrightarrow W$. Using the conditions above we see that

$$(g \circ f) \circ \pi = g \circ (f \circ \pi) = g \circ \pi' = \pi.$$

But, the identity map $\mathrm{Id}_W \colon W \longrightarrow W$ is also a morphism from W to W such that $\mathrm{Id}_W \circ \pi = \pi$. Thus both $g \circ f$ and Id_W are solutions to the problem "find a morphism $p \colon W \longrightarrow W$ so that $p \circ \pi = \pi$. By the universal property of W, there is a unique such map. Therefore $g \circ f = \mathrm{Id}_W$.

Similarly,

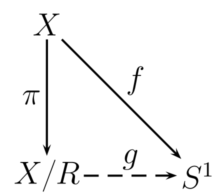
$$(f \circ g) \circ \pi' = f \circ (g \circ \pi') = f \circ \pi = \pi'.$$

Since we also have $\mathrm{Id}_{W'} \circ \pi' = \pi'$, we conclude using the universal property of W' that $f \circ g = \mathrm{Id}_{W'}$.

2.

(a) Since each of the maps $f_1(x) = \cos(2\pi x)$ and $f_2(x) = \sin(2\pi x)$ are continuous maps from $\mathbb{R}$ to $\mathbb{R}$, each are also continuous maps from the subspace $[0,1] \subseteq \mathbb{R}$ to $\mathbb{R}$. The universal property of the product then guarantees that the map $f\colon [0,1] \longrightarrow \mathbb{R}^2$ given by $f(x) = (f_1(x), f_2(x))$ is continuous. The image of f lies in $S^1 \subseteq \mathbb{R}^2$, and so by the universal property of a subspace, the corresponding map $[0,1] \longrightarrow S^1$ obtained by restricting the codomain (a map we continue to call f) is also continuous.

(b) Under this map $f(0) = f(1)$, so by the universal property of the quotient there is a unique continuous map $g\colon X/R \longrightarrow S^1$ so that $f = g \circ \pi$.



The surjectivity of g follows from the surjectivity of f and the factorization above. To see the injectivity of g we need to check that $f(x_1) = f(x_2) \implies x_1 \sim_R x_2$. (The direction $x_1 \sim_R x_2 \implies f(x_1) = f(x_2)$ was what we needed to check to use the universal property of X/R to get the morphism g.)

But f is injective when restricted to $(0,1)$, and no point in $(0,1)$ is sent to $f(0) = (1,0) = f(1) \in S^1$, So for $x_1 \in (0,1)$ and $x_2 \in [0,1]$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. The only remaining points $\{0,1\}$, are equivalent under R. Thus g is also injective.

(c) Since $g\colon X/R \longrightarrow S^1$ is a continuous map, τ_Q (the topology on X/R) is finer than τ_S (the topology on S^1).

(d) A base for the topology on $[0,1]$ is

$$\mathcal{B} = \Big\{(a,b) \,\Big|\, 0 < a < b < 1\Big\} \bigcup \Big\{[0,b) \,\Big|\, 0 < b < 1\Big\} \bigcup \Big\{(a,1] \,\Big|\, 0 < a < 1\Big\}.$$

Open sets of the form (a,b) are already R-saturated. But, if an open set $U \subset [0,1]$ contains an element of the base of the form $[0,b)$ then to be R-saturated U must also contain an element of the base of the form $(a,1]$, and vice-versa. This is the only condition for an open set $U \subseteq [0,1]$ to be R-saturated (for this relation R).

From this we conclude that a base for τ_Q (in terms of the corresponding R-saturated subsets) is

$$\mathcal{B}_R = \Big\{(a,b) \,\Big|\, 0 < a < b < 1\Big\} \bigcup \Big\{[0,b) \cup (a,1] \,\Big|\, a,\, b \in (0,1)\Big\}$$

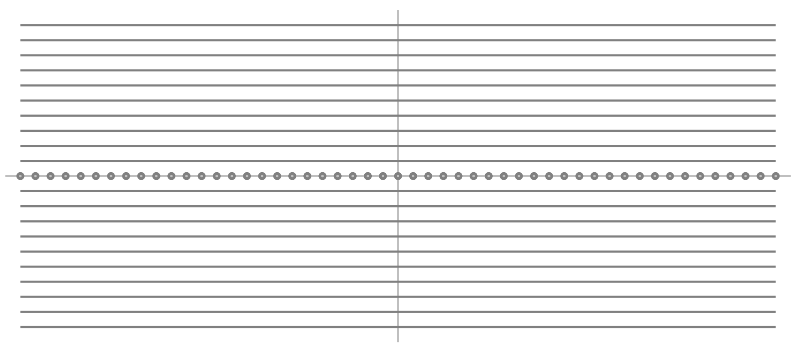
Each of these elements of the base can be realized as pullbacks of open sets in τ_S. Intervals of the form (a,b) are $g^{-1}((2\pi a,\, 2\pi b))$, where (θ_1, θ_2) refers to the "angular interval" in S^1 consisting of those points $(x, y) \in S^1$ whose angles lie between θ_1 and θ_2) (modulo 2π).

An element of the base of the form $[0, b) \cup (a, 1]$, with $a \geqslant b$, is the pullback $g^{-1}((2\pi\, a, 2\pi b + 2\pi))$. If $a < b$ then $[0, b) \cup (a, 1] = [0, 1]$, which is the pullback of the open set S^1.

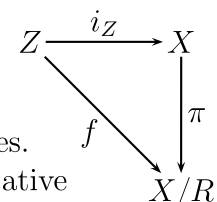
Thus every element of a base for τ_Q is the pullback of an element in τ_S, and so both topologies are the same.

3.

(a) The equivalence classes are the points $\{(x, 0)\}$, $x \in \mathbb{R}$ on the x-axis (each in their own equivalence class), and the horizontal lines $y = b$ for each $b \neq 0$, also distinct equivalence classes if $b_1 \neq b_2$.



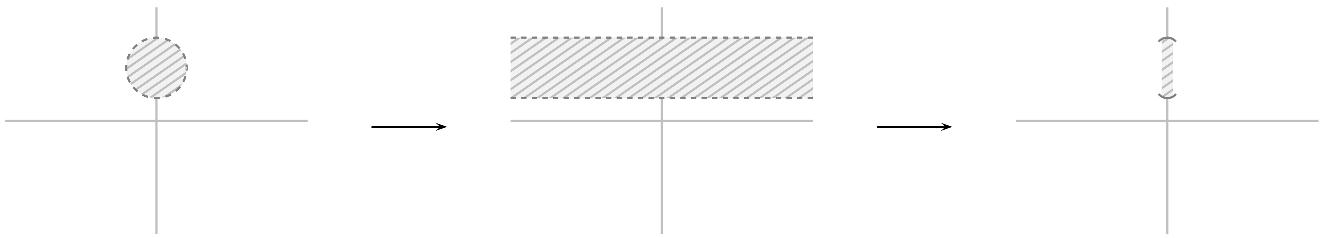
(b) The diagram of maps is shown at right. the composition $f = \pi \circ i_Z$ is continuous since each of i_Z and π are continuous.



Each point of Z meets one and only one of the equivalence classes. Each equivalence class of the form $y = b$ ($b \neq 0$) has unique representative $(0, b) \in Z$, and each equivalence class $[(x, 0)]$ (consisting of the single point $(x, 0)$) is contained in Z. Thus the map f is both surjective and injective.

(c) Considering the set $W \subseteq \mathbb{R}^2$, its R-saturation is $\{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$. This is an open set, but not a closed set. The closure of this set is all of $\mathbb{R}^2$. From this we conclude that the only R-saturated closed subset of $\mathbb{R}^2$ containing W is all of $\mathbb{R}^2$, and therefore (under our correspondence between R-saturated subsets of $\mathbb{R}^2$ and subsets of X/R), the closure of W in X/R is X/R. (Or, under the bijection of Z and X/R, the closure of W in Z is all of Z.)

(d) Here is a picture of the open ball we start with, its R-saturation (which is open in $\mathbb{R}^2$), and the restriction of the R-saturation to Z :

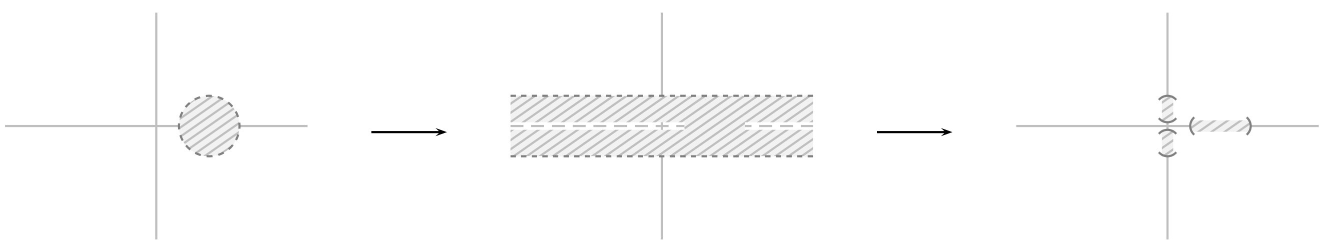


If the open ball is centered at $(0, y_0)$, then the process above corresponds to the sets

$$\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \sqrt{x^2 + (y - y_0)^2} < \delta\right\} \longrightarrow \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, |y - y_0| < \delta\right\} \longrightarrow \left\{(0, y) \in \mathbb{R}^2 \,\middle|\, |y - y_0| < \delta\right\}.$$

(e) Thus, for any $(0, y_0) \in Z$, $y_0 \neq 0$, and any δ such that $0 < \delta < |y|$, the interval $\{(0, y) \mid |y - y_0| < \delta\}$ is an open set in τ_Q. In particular, this implies that any open set (in the standard topology) on the y-axis not containing $(0, 0)$ is also an open set in τ_Q.

(f) Here is a picture of the same process as in (d), first starting with an open ball centered on the x-axis which does not contain the origin.



If the open ball is centered at $(x_0, 0)$, then the process above corresponds to the sets

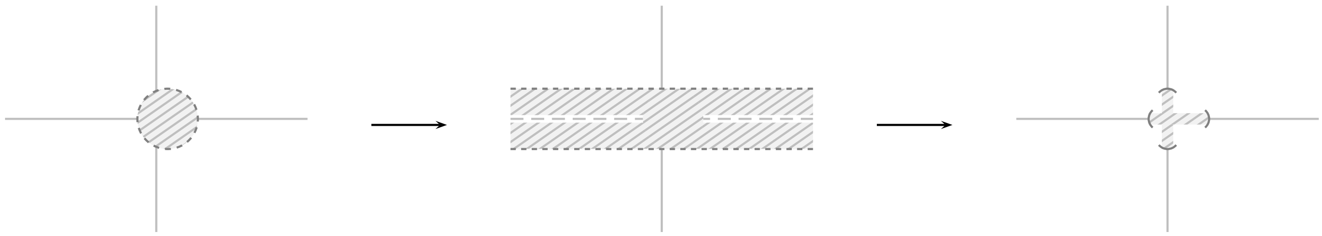
$$\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \sqrt{(x - x_0)^2 + y^2} < \delta\right\} \longrightarrow \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 0 < |y| < \delta\right\} \cup \left\{(x, 0) \,\middle|\, |x - x_0| < \delta\right\}$$

$$\longrightarrow V_\delta \cup \left\{(x, 0) \in \mathbb{R}^2 \,\middle|\, |x - x_0| < \delta\right\},$$

where, as in the assignment,

$$V_\delta = \left\{(0, y) \,\middle|\, 0 < |y| < \delta\right\}.$$

If the open ball does contain the origin, then the picture is largely the same, with the difference that $(0, 0)$ is now in the resulting open subset of Z, and the subset is connected.



In this case the restriction to Z again has the form

$$V_\delta \cup \left\{(x,0) \in \mathbb{R}^2 \,\middle|\, |x - x_0| < \delta\right\}.$$

(g) The open sets appearing in (e) and (f) form a base for the topology τ_Q (because they form a base for the R-saturated subsets of $\mathbb{R}^2$). The open sets in (e) contain no point on the x-axis, while the open sets in (f), each contain a subset of the form V_δ with $\delta > 0$.

Thus, if $U \subseteq Z$ is open in τ_Q, and contains a point on the x-axis, it must contain an element of the base of the type from (f), and so contain a subset of the form V_δ for some $\delta > 0$.

(h) An open set $U \in \tau_S$ is in τ_Q if and only if it passes the test above : If U contains a point on the x-axis, then U contains a subset of the form V_δ, for some $\delta > 0$.

The condition is obviously necessary, in light of (g). Suppose that this condition is satisfied. If U does not contain any point of the x-axis, then U is a union of open intervals on the y-axis, and hence in τ_Q by (e).

If U does contain points of the x-axis, let $\delta > 0$ be such that $V_\delta \subset U$. The restriction of U to the x-axis is a union of intervals, each one of which can be written as a union of intervals of length $\leqslant \delta$, and hence a union of sets of the form in (f), each one with an associated $V_{\delta'}$ with $\delta' \leqslant \delta$ and so $V_{\delta'} \subseteq V_\delta$.

Therefore the union of these sets is still contained in U, and covers all points of U on the x-axis. The remaining points of U on the y-axis can be covered by open intervals not containing $(0,0)$, open sets which, by (e), are in τ_Q. Therefore U can be written as a union of sets in τ_Q, and so is itself in τ_Q.