1.

(a) Since A is the finite union $A = \bigcup_{i=1}^{n}\{x_i\}$, and since each $\{x_i\}$ is closed, A is closed.

(b) Let x_i be a point A, and $B = \complement_A\{x_i\}$. Then, by (a), B is closed in X, and so $B \cap A = B$ is closed in A. Therefore each subset $\{x_i\}$ is open in A, and so A has the discrete topology.

(c)

- For each $i = 1,\ldots, n$, $U_i = U_i' \cap V_i$ is the intersection of two open sets, and hence open, while $U_{n+1} = \bigcap_{i=1}^{n} W_i$ is the intersection of finitely many open sets and so also open.
- For each $i = 1,\ldots, n$, $x_i \in U_i'$ and $x_i \in V_i$, so $x_i \in U_i' \cap V_i$. Similarly for $i = n+1$, $x_{n+1} \in W_j$ for each $j = 1,\ldots, n$, so $x_{n+1} \in \bigcap_{j=1}^{n+1} W_j = U_{n+1}$.
- If $i, j \in \{1, \ldots, n\}$ with $i \neq j$ then

  $$U_i \cap U_j = (U_i' \cap V_i) \cap \left(U_j' \cap V_j\right) = \left(U_i' \cap U_j'\right) \cap V_i \cap V_j = \varnothing \cap V_i \cap V_j = \varnothing,$$

  since $U_i' \cap U_j' = \varnothing$ if $i \neq j$.

  On the other hand, if $j = n+1$ and $i \in \{1, \ldots, n\}$ then

  $$U_i \cap U_{n+1} = (U_i' \cap V_i) \cap \left(\bigcap_{k=1}^{n} W_k\right) = U_i' \cap (V_i \cap W_i) \cap \left(\bigcap_{\substack{k\in\{1,\ldots,n\}\\ k\neq i}} W_i\right) = \varnothing,$$

  since $V_i \cap W_i = \varnothing$.

This establishes the properties we were claiming.

(d) The proof is by induction.

Base Case $n = 2$: This is the definition of a Hausdorff space. Given $x_1, x_2 \in X$, $x_1 \neq x_2$ we can find open sets U_1 and U_2 such that $x_1 \in U_1$ and $x_2 \in U_2$ and $U_1 \cap U_2 = \varnothing$.

Inductive Step : Given distinct $x_1,\ldots, x_n \in X$, by induction we may assume we have open sets $U_1', \ldots, U_n'$ such that $x_i \in U_i'$ for each i, and $U_i' \cap U_j' = \varnothing$ if $i \neq j$.

Given the point x_{n+1} (distinct from $x_1,\ldots, x_n$), since X is Hausdorff for each $i = 1$, $\ldots$ n there are open sets separating x_{n+1} and each x_i. I.e., for each i there are open sets V_i and W_i with $x_i \in V_i$, $x_{n+1} \in W_i$, and $W_i \cap V_i = \varnothing$.

Applying the construction of part (c) we obtain open sets $U_1, \ldots, U_{n+1}$ with $x_i \in U_i$ for each $i = 1, \ldots, n+1$, and $U_i \cap U_j$ if $i \neq j$, completing the inductive step.

2.

(a) Suppose that (x_1, y_1) and (x_2, y_2) are two distinct points of $X \times Y$. Then either $x_1 \neq x_2$, or $y_1 \neq y_2$, or both. Suppose that $x_1 \neq x_2$. Then since X is Hausdorff there are open sets U_1, $U_2 \subseteq X$ with $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \varnothing$.

Then $U_1 \times Y$ and $U_2 \times Y$ are open sets of $X \times Y$, $(x_1, y_1) \in U_1 \times Y$, $(x_2, y_2) \in U_2 \times Y$, and

$$(U_1 \cap Y) \cap (U_2 \cap Y) = (U_1 \cap U_2) \times Y = \varnothing.$$

The remaining case is that $x_1 = x_2$ but $y_1 \neq y_2$, in which case a similar argument, pulling back open sets from Y, works.

(b) Since $\Delta_X = \{(x, x) \mid x \in X\} \subseteq X \times X$, and $q_1(x_1, y_1, x_2, y_2) = (x_1, x_2)$,

$$q_1^{-1}(\Delta_X) = \left\{(x_1, y_2, x_2, y_2) \,\middle|\, q(x_1, y_1, x_2, y) \in \Delta_X\right\} = \left\{(x, y_1, x, y_2) \,\middle|\, x \in X,\ y_1,\ y_2 \in Y\right\}.$$

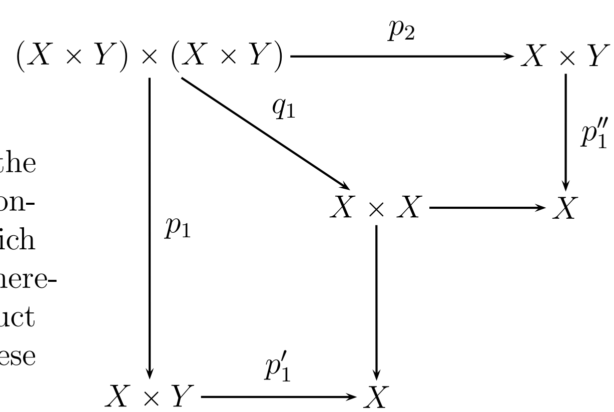
Similarly

$$q_2^{-1}(\Delta_Y) = \left\{(x_1, y, x_2, y) \,\middle|\, x_1,\ x_2 \in X,\ y \in Y\right\}.$$

I.e., $q_1^{-1}(\Delta_X)$ is the subset of $X \times Y \times X \times Y$ where the first and third coordinates are equal,and $q_2^{-1}(\Delta_Y)$ is the subset where the second and fourth coordinates are equal. Therefore

$$q_1^{-1}(\Delta_X) \cap q_2^{-1}(\Delta_Y) = \left\{(x, y, x, y) \,\middle|\, x \in X,\ y \in Y\right\}$$

which is (under the identification $(X \times Y) \times (X \times Y) = X \times Y \times X \times Y$) the diagonal $\Delta_{X \times Y}$.



(c) We note that each of q_1 and q_2 are continuous maps.

For instance, the composition $p_1' \circ p_1$ is the map sending (x_1, y_1, x_2, y_2) to x_1, and is continuous, and the composition $p_1'' \circ p_2$, which sends (x_1, y_1, x_2, y_2) to x_2 is continuous. Therefore (by the universal property of the product $X \times X$ ...), the map q_1 obtained from these maps is continuous.

Similarly q_2 is continuous.

If X and Y are Hausdorff then Δ_X and Δ_Y are closed in $X \times X$ and $Y \times Y$ respectively. Since q_1 and q_2 are continuous, $q_1^{-1}(\Delta_X)$ and $q_2^{-1}(\Delta_Y)$ are then closed in $(X \times Y) \times (X \times Y)$.

Since, by (b), $\Delta_{X\times Y} = q_1^{-1}(\Delta_X) \cap q_2^{-1}(\Delta_Y)$, $\Delta_{X\times Y}$ is closed in the product, and therefore $X \times Y$ is Hausdorff.

3.

(a) In **H6 Q3** we have seen that for any point $(0, y_1) \in X/R$ with $y_1 \neq 0$, and any $\delta < |y_1|$, that the set $\{(0, y) \mid |y - y_1| < \delta\}$ is an open set in X/R. That is, away from $(0, 0)$, the topology of X/R "on the y-axis" is just like the topology of the y-axis in $\mathbb{R}^2$, and so the usual arguments work to establish the Hausdorff property.

Given y_1, y_2, with $y_1 \neq y_2$, pick δ to be strictly less than $\min(|y_1|, |y_2|, \frac{|y_2 - y_1|}{2})$. Then the sets

$$U_1 = \{(0, y) \mid |y - y_1| < \delta\} \quad \text{and} \quad U_2 = \{(0, y) \mid |y - y_2| < \delta\}$$

are both open in X/R, and satisfy $y_1 \in U_1$, $y_2 \in U_2$, and $U_1 \cap U_2 = \varnothing$.

(b) As in **H6 Q3**, for each $\delta > 0$ let

$$V_\delta = \left\{(0, y) \,\middle|\, 0 < |y| < \delta\right\},$$

considered (via the quotient map) as a set in X/R. In the previous homework assignment we have seen that for any point $(x_1, 0) \in X/R$, and any $\delta > 0$ that

$$\{(x, 0) \mid |x - x_1| < \delta\} \cup V_\delta$$

is an open set in X/R.

Given $(x_1, 0)$, $(0, y_2) \in X/R$, with $y_2 \neq 0$, let $\delta = \frac{|y_2|}{3}$. Then both

$$U_1 = \{(x, 0) \mid |x - x_1| < \delta\} \cup V_\delta$$

and

$$U_2 = \{(0, y) \mid |y - y_2| < \delta\}$$

are open sets of X/R. By construction $x_1 \in U_1$ and $y_2 \in U_2$. Finally $U_1 \cap U_2 = \varnothing$, since by choice of δ we know that points of U_2 are at least distance 2δ from $(0, 0)$, while points of U_1 are within distance δ of $(0, 0)$, and so the two sets cannot meet.

(c) However, given $(x_1, 0)$ and $(x_2, 0)$, if U_1 and U_2 are open sets of X/R containing $(x_1, 0)$ and $(x_2, 0)$ respectively, then by **H6 Q3**(b), there is a $\delta_1 > 0$ such that $V_{\delta_1} \subseteq U_1$,and a $\delta_2 > 0$ such that $V_{\delta_2} \subseteq U_2$. Therefore $U_1 \cap U_2$ contains at least V_δ, where $\delta = \min(\delta_1, \delta_2)$. In particular it is impossible to separate $(x_1, 0)$ and $(x_2, 0)$ by open sets of X/R.

4.

(a) The map π sends $x \in X$ to $[x]$ (the equivalence class of x) in X/R. Therefore

$$(\pi\times\pi)^{-1}(\Delta_{X/R}) = \left\{(x_1, x_2) \in X \times X \,\middle|\, [x_1] = [x_2]\right\} = \left\{(x_1, x_2) \in X \times X \,\middle|\, x_1 \sim_R x_2\right\} = \Gamma_R.$$

(b) If X/R is Hausdorff, then $\Delta_{X/R}$ is closed in $(X/R) \times (X/R)$, and therefore $\Gamma_R = (\pi \times \pi)^{-1}(\Delta_{X/R})$ is closed in $X \times X$.

(c) Let us first suppose that there is a $t \in \mathbb{R}_{>0}$ such that $(x_2,\, y_2) = (tx_1,\, ty_1)$ then

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ tx_1 & ty_1 \end{vmatrix} = 0$$

(for instance, since the rows are linearly dependent), $x_1x_2 = t(x_1)^2 \geqslant 0$, and $y_1y_2 = t(y_1)^2 \geqslant 0$.

Conversely, suppose that (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2\backslash\{(0,0)\}$, and that

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0.$$

Since the determinant is zero, the rows are linearly dependent. Linear dependence for two vectors means that one is a scalar multiple of the other (including the possibility that the scalar is zero). But, since neither vector is zero, the scalar cannot be zero, and thus each is a non-zero scalar multiple of the other. In particular, there is a $t \in \mathbb{R}^*$ so that $(x_2, y_2) = t(x_1, y_1)$.

If we now assume the other two conditions $x_1x_2 = t(x_1)^2 \geqslant 0$, and $y_1y_2 = t(y_1)^2 \geqslant 0$, then adding them gives $t(x_1^2 + y_1^2) \geqslant 0$. Since $(x_1, y_1) \neq (0, 0)$, $x_1^2 + y_1^2 > 0$, and so we can divide and conclude that $t \geqslant 0$. Thus, there is $t \in \mathbb{R}_{>0}$ so that $(x_2, y_2) = t(x_1, y_1)$.

This shows the equivalence of the given algebraic conditions and R-equivalence in this case.

(d) The subset of $X \times X$ (where $X = \mathbb{R}^2\backslash\{(0,0)\}$) described by the conditions in (c) is a closed subset. Each of the maps

$$\begin{array}{llll} f: & X \times X & \longrightarrow & \mathbb{R} \\ & (x_1, y_1, x_2, y_2) & \mapsto & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \end{array}$$

$$\begin{array}{llll} g: & X \times X & \longrightarrow & \mathbb{R} \\ & (x_1, y_1, x_2, y_2) & \mapsto & x_1x_2 \end{array}$$

$$\begin{array}{llll} h: & X \times X & \longrightarrow & \mathbb{R} \\ & (x_1, y_1, x_2, y_2) & \mapsto & y_1y_2 \end{array}$$

is continuous. (For instance, each of the coordinate projection maps are continuous, and then one can use the result of **H4 Q1**(g), applied to $X \times X$.)

Thus each of the sets $f^{-1}(\{0\})$, $g^{-1}([0,\infty))$, and $h^{-1}([0,\infty))$ are closed subsets of $X \times X$, since $\{0\}$ and $[0,\infty)$ are closed subsets of $\mathbb{R}$. The subsets $f^{-1}(\{0\})$, $g^{-1}([0,\infty))$, and $h^{-1}([0,\infty))$ are the solutions to the individual conditions in (c). By (c) the intersection of these conditions is $\Gamma_{\mathbb{R}}$. Therefore $\Gamma_{\mathbb{R}}$ is a closed subset of $X \times X$.

(e) The other piece of $\Gamma_{\mathbb{R}}$ is the subset

$$\left\{(x_1, y, x_2, y) \,\middle|\, x_1,\ x_2, y \in \mathbb{R},\ y \neq 0\right\}.$$

Letting $q_2 \colon \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be the map $q_2(x_1, y_1, x_2, y_2) = (y_1, y_2)$ as in **Q2**, the subset above is $q_2^{-1}(\Delta_{\mathbb{R}}) \cap \{(x_1, y_1, x_2, y_2) \mid y_1,\ y_2 > 0\}$. That is, the subset above is the intersection of a closed set and an open set.

(f) In this case the subset Γ_R is not closed. Let $x_1,\ x_2 \in \mathbb{R}$ be any two numbers with $x_1 \neq x_2$. Then for all $n \geqslant 1$, $n \in \mathbb{N}$, the point $(x_1, \frac{1}{n}, x_2, \frac{1}{n})$ is in the set in (e) above, and so in Γ_R. The limit of this sequence of points (as $n \to \infty$) is $(x_1, 0, x_2, 0)$. This is not in the piece from (e) (since the y-coordinates are zero), and is not in the other piece of Γ_R since $x_1 \neq x_2$. Thus, when $x_1 \neq x_2$, $(x_1, 0, x_2, 0)$ is a point in $\overline{\Gamma}_R$ which is not in Γ_R. Thus Γ_R is not closed in $X \times X$.