1.

(a) C_{x_0} is the union of connected sets with at least one point in common, so by the proposition from class C_{x_0} is connected.

(b) If A is connected set containing x_0, then A is in the list of sets we take a union of when constructing C_{x_0}, so $A \subseteq C_{x_0}$.

(c) We have $C_{x_0} \subseteq \overline{C}_{x_0}$. Since C_{x_0} is connected its closure $\overline{C}_{x_0}$ is connected. Since the closure contains x_0, by the maximalitity in (b) we also have $\overline{C}_{x_0} \subseteq C_{x_0}$. Therefore $\overline{C}_{x_0} = C_{x_0}$, and C_{x_0} is closed.

(d) If $A \cap C_{x_0} \neq \varnothing$, with A connected, then $A \cup C_{x_0}$ is a connected subset (since both A and C_{x_0} are connected, and have a nonempty intersection) and so by the maximality in (b), we have $A \cup C_{x_0} \subseteq C_{x_0}$. Since $A \subseteq A \cup C_{x_0}$, this gives $A \subseteq C_{x_0}$.

(e) If $C_{x_0} \cap C_{x_1} \neq \varnothing$, then by (d) applied to C_{x_0} we have $C_{x_1} \subseteq C_{x_0}$. On the other hand, applying (d) to C_{x_1} gives $C_{x_0} \subseteq C_{x_1}$. Therefore $C_{x_0} = C_{x_1}$.

(f) The equivalence classes of R are the connected components, and by (c) each equivalence class is closed. Therefore each point of X/R is closed, and X/R is a T_1 space.

(g) Suppose we show that each closed subset $B \subseteq X/R$ with two or more points is disconnected. Then, given a subset $A \subseteq X/R$ with two or more points, if A were connected, then $\overline{A}$ would be a closed connected set with two or more points. We are assuming that we can show that no such thing exists, and therefore A (with its two or more points) cannot be connected.

(h) Let C be a connected component contained in D. Since D contains at least two connected components, D is strictly larger (in terms of containment) than C. If D were connected it would contradict the maximality of C (i.e, the property in part (b)).

(i) By the definition of the subspace topology, if D_1 and D_2 are closed in D, then there are closed sets Z_1 and $Z_2 \subseteq X$ so that $D_1 = D \cap Z_1$ and $D_2 = D \cap Z_2$. But, since D is closed in X, this means that D_1 and D_2 are also closed subsets of X.

(j) From (i), and the definition of the D_i, we have that $D = D_1 \sqcup D_2$, and that D_1 and D_2 are closed. Therefore, for any $C \subseteq D$, $(C \cap D_1)$ and $(C \cap D_2)$ split C into two disjoint closed sets. Since C is connected, one of those sets must be all of C and the other the empty set. Therefore either $C \subseteq D_1$ or $C \subseteq D_2$.

(k) Since $D = \pi^{-1}(B)$, $D_1 = \pi^{-1}(B_1)$, and $D_2 = \pi^{-1}(B_2)$,

- $D = D_1 \cup D_2$ means $\pi^{-1}(B) = \pi^{-1}(B_1) \cup \pi^{-1}(B_2) = \pi^{-1}(B_1 \cup B_2)$ and so $B = \pi(\pi^{-1}(B)) = \pi(\pi^{-1}(B_1 \cup B_2)) = B_1 \cup B_2$.

- $\varnothing = D_1 \cap D_2$ means $\pi^{-1}(\varnothing) = \pi^{-1}(B_1) \cap \pi^{-1}(B_2) = \pi^{-1}(B_1 \cap B_2)$ and so $\varnothing = \pi(\varnothing) = \pi(\pi^{-1}(B_1 \cap B_2)) = B_1 \cap B_2$.

We also know neither B_1 nor B_2 is empty, since D_1 and D_2 are nonempty. Therefore B_1 and B_2 give a decomposition of B into disjoint nonempty closed subsets, so B is disconnected.

2.

(a) From class we know that if $f\colon X \longrightarrow Y$ is a continuous map, and $A \subseteq X$ quasi-compact, then $f(A)$ is quasi-compact. I.e., the image of a quasi-compact set is quasi-compact. The quotient map $\pi\colon X \longrightarrow X/R$ is surjective. Therefore if X is quasi-compact, its image $\pi(X) = X/R$ is quasi-compact.

(b) S_1 is a subset of $\mathbb{R}^2$ and has the subspace topology. Since $\mathbb{R}^2$ is Hausdorff, S_1 is Hausdorff.

(c) The interval $[0,1]$ is quasi-compact, and therefore by (a) $S^1 = [0,1]/R$ is quasi-compact. Since g is a continuous bijection from a quasi-compact space to a Hausdorff space, g is a homeomorphism by the theorem from class.

3.

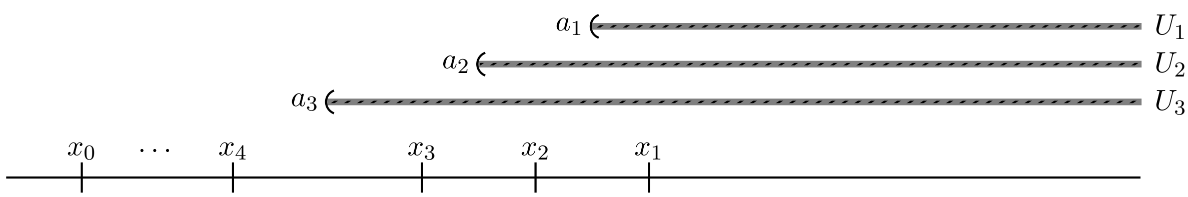
(a) Suppose that $\{U_i\}_{i \in I}$ is a collection of open subsets in $\mathbb{R}$ (with the "arrow" topology) which cover A. Each U_i is of the form (a_i, ∞) for some $a_i \in \mathbb{R}$. Since $x_0 \in A$, one of the sets must contain x_0, so there is an i such that $a_i < x_0$. But since $x_0 = \inf A$,

$$A \subseteq [x_0, \infty) \subseteq (a_i, \infty) = U_i,$$

and therefore the set U_i alone covers all of A.

(b) Suppose that $x_0 \notin A$. Then there is a decreasing sequence $x_1 > x_2 > x_3 > \cdots$, with each $x_i \in A$ such that $x_0 = \lim_{n \to \infty} x_n$. For each $i \geqslant 1$, set $a_i = \frac{x_i + x_{i+1}}{2}$, and $U_i = (a_i, \infty)$. Then

- The sets $\{U_i\}_{i\geqslant 1}$ cover A. Since $\lim_{n\to\infty} x_n = \inf A$, and since $\inf A \notin A$, for every $z \in A$ there is an i so that $x_i < z$. But then $x_{i+1} < z$ too, and so $a_i < z$, and therefore $z \in U_i$.
- The cover $\{U_i\}_{i\geqslant 1}$ has no finite subcover. Suppose $J \subset \mathbb{N}_{>0}$ is a finite set such that the U_i, $i \in J$ cover A. Set $j = \max J$. Then $x_{j+1} < a_j$. Since the $\{a_i\}$ form a decreasing sequence, this means that $x_{j+1} < a_i$ for all $i \in J$. But then $x_{j+1} \notin \bigcup_{i\in J} U_i$, so this finite subset does not cover A.



(c) The closed sets in the arrow topology are those of the form $(-\infty, a]$. In particular, every closed set in this topology is bounded from above. Thus the set $A = [0, \infty)$ is not closed in this topology. Since A contains its own infimum (namely 0), it is quasi-compact in the arrow topology.

A similar example is to take a finite interval like $A = [0, 1]$. Since A contains its own infimum, A is quasi-compact in the subspace topology of the arrow topology. The set A is also not closed, since closed sets in the arrow topology are unbounded to the left.

(d) Taking $A = [0, \infty)$ as above $B = [1, \infty)$ is a subset of A which, since it is not bounded from above, is not the intersection of A with any $(-\infty, a]$. Therefore B is not closed in the topology of A. Since B contains its own infimum, B is quasi-compact in the arrow topology.