Math 347 Answers for Homework 9

1.

(a) Any compact subspace of a Hausdorff space is closed.

(b) Each coordinate function f_i is continuous. Since A is compact, by the generalized extremal value theorem f_i acheives a minimum and maximum on A. Letting a_i be the minimum, and b_i the maximum, $a_i \leqslant f_i(x) \leqslant b_i$ for all $x \in A$, or equivalently, $f_i(A) \in [a_i, b_i]$.

(c) One possible meaning for "bounded" is that there exists an $R > 0$ so that $\|x\| \leqslant R$ for all $x \in A$, where for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Another possible meaning for "bounded" is that there is an $R > 0$ so that A is contained in a rectangle of side length $2R$ centered at the origin, that is, so that A is contained in $[-R, R] \times [-R, R] \times \cdots \times [-R, R]$.

From part (b) we know that the i-th coordinate of points of A lie in the interval $[a_i, b_i]$, and so $A \subseteq [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Setting $r = \max(|a_1|, |b_1|, |a_2|, |b_2|, \ldots, |a_n|, |b_n|)$, and $R = r\sqrt{n}$, for each $x \in A$ the value of the absolute value of the i-th coordinate is $\leqslant r$, so

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \leqslant \sqrt{r^2 + r^2 + \cdots + r^2} = \sqrt{nr^2} = r\sqrt{n} = R,$$

and so A is bounded as in the first definition.

Or, defining r as above then $A \subseteq [-r, r] \times [-r, r] \times \cdots \times [-r, r]$, and so A is bounded as in the second definition.

(d) Now we are supposing that A is bounded. If we are using the first definition, there is an R so that $\|x\| \leqslant R$ for all $x \in A$. But, the solid ball of radius R centered at the origin is contained in the product of intervals $[-R, R] \times [-R, R] \times \cdots \times [-R, R]$, and so A, being contained in this solid ball, is also contained in this product of intervals, so that we can take $a_i = -R$ and $b_i = R$ for $i = 1, \ldots, n$.

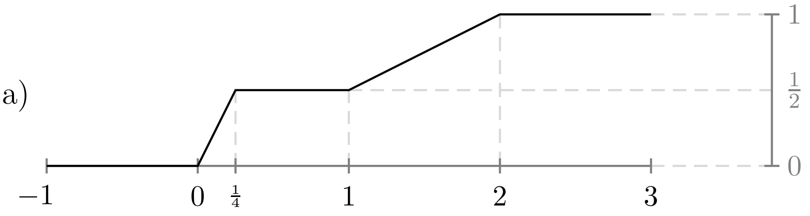
If we are using the second definition of bounded, then there is an R so that $A \subseteq [-R, R] \times [-R, R] \times \cdot \times [-R, R]$, and we are again done (and we can again take $a_i = -R$ and $b_i = R$ for all i).

(e) We have shown in class that a closed subset of a compact space is again compact. So, if $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact, and A a closed subset of $[a_1, b_1] \times \cdots \times [a_n, b_n]$ then A is compact. Thus it suffices to show that this product of closed intervals is compact.

(f) We have seen in class that each closed interval $[a,b] \subseteq \mathbb{R}$ is compact. Since products of compact topological spaces are again compact, $[a_1,b_1] \times \cdots \times [a_n,b_n]$ is compact.

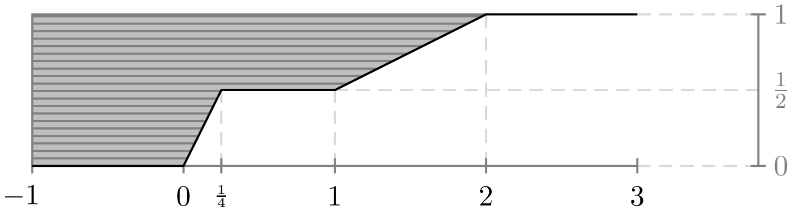
2.

(a)



$$f(x) = \begin{cases} 0 & \text{if } x \in [-1,0] \\ 2x & \text{if } x \in [0,\frac{1}{4}] \\ \frac{1}{2} & \text{if } x \in [\frac{1}{4},1] \\ \frac{x+1}{2} & \text{if } x \in [1,2] \\ 1 & \text{if } x \in [2,3] \end{cases}$$

One can see the values of the function directly from the definition. Another way is to consider t to be the height (i.e., on the y-axis) and for each $t \in [0,1]$ draw the interval U_t sideways at height t :



$t \in (\frac{1}{2},1]$, so draw interval $[-1,2t)$

$t = \frac{1}{2}$, so draw interval $[-1,\frac{1}{2})$

$t \in [0,\frac{1}{2})$, so draw interval $[-1,\frac{t}{2})$

The "lower hull" of this picture is the graph of f.

(b) $V_{\frac{1}{2}} = \bigcup_{\epsilon>0} U_{\frac{1}{2}-\epsilon} = \bigcup_{\epsilon>0} [-1, \frac{\frac{1}{2}-\epsilon}{2}) = [-1,\frac{1}{4})$.

(c) Yes, as we can see from the graph above, $f(x)$ is an increasing function and only reaches the value of $\frac{1}{2}$ when $t = \frac{1}{4}$. So, on the interval $[-1,\frac{1}{4})$, $f(x) < \frac{1}{2}$.

(d) $U_{\frac{1}{2}} = [-1,\frac{1}{2})$, so $U_{\frac{1}{2}} \backslash V_{\frac{1}{2}} = [\frac{1}{4},\frac{1}{2})$.

(e) On the interval $[\frac{1}{4},\frac{1}{2})$, f is constant, equal to $\frac{1}{2}$.

(f) In this example, the set $\{x \in [-1,3] \mid f(x) < \frac{1}{2}\} = V_{\frac{1}{2}}$. We clearly can't allow any of the points in (d), since then f is equal to $\frac{1}{2}$.

(g) $Z_{\frac{1}{2}} = \bigcap_{\epsilon>0} \overline{U}_{\frac{1}{2}+\epsilon} = \bigcap_{\epsilon>0} [-1, 2(\frac{1}{2}+\epsilon)] = \bigcap_{\epsilon>0} [-1, 1+2\epsilon] = [-1,1]$.

(h) Yes, the interval $[-1, 1]$ is exactly the set of points where $f(x) \leqslant \frac{1}{2}$.

(i) $\overline{U}_{\frac{1}{2}} = [-1, \frac{1}{2}]$, so $Z_{\frac{1}{2}} \backslash \overline{U}_{\frac{1}{2}} = (\frac{1}{2}, 1]$.

(j) On the interval $(\frac{1}{2}, 1]$, f is again constant, equal to $\frac{1}{2}$.

(k) The set of points where $f(x) > \frac{1}{2}$ is the interval $(1, 3]$. This is exactly the complement of $Z_{\frac{1}{2}} = [-1, 1]$ in $X = [-1, 3]$.