1.

(a) Let a_1 and a_2 be two distinct elements of $\mathbb{Z}$. Pick n larger than $|a_1 - a_2|$ (for instance, $n = |a_1 - a_2| + 1$). Then $a_1 \not\equiv a_2$ mod n, and so $U = B_n(a_1)$ and $V = B_n(a_2)$ are disjoint open sets with $a_1 \in U$ and $a_2 \in V$. Thus X is Hausdorff.

(b) Suppose that $A \subseteq \mathbb{Z}$ is a subset with at least two distinct elements, say a_1 and a_2. Again pick n so that $n > |a_1 - a_2|$. Then $B_n(a_1)$ is both open and closed in $\mathbb{Z}$, so $A_1 := B_n(a_1) \cap A$ is a subset of A which is both open and closed in A. The subset A_1 is not empty, since $a_1 \in A_1$, and is not all of A since $a_2 \notin A_1$. Therefore A_1 and $\complement_A A_1$ disconnect A.

This shows that no subset of $\mathbb{Z}$ with two or more elements is connected. Therefore $\mathbb{Z}$ is totally disconnected.

In parts (c) and (d), two solutions for each problem are given. One which is more "down to earth", and one using the diagram idea suggested in the problem after part (d). They are both really the same solution, although it may not look like that at first.

(c) "Down to earth" solution. Fix $n \geqslant 1$ and an $a \in \mathbb{Z}$. We want to show that $A^{-1}(B_n(a))$ is an open set. A point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is in $A^{-1}(B_n(a))$ exactly when $x + y \equiv a \,(\text{mod } n)$. Let $\overline{a}$ be the class of a mod n. Since $\mathbb{Z}/n\mathbb{Z}$ is finite, there are only finitely many solutions to the equation $x + y = \overline{a} \,(\text{mod } n)$. Let $(\overline{b}_1, \overline{c}_1)$,..., $(\overline{b}_n, \overline{c}_k)$ be the finitely many solutions. Then for each pair (x, y) such that $x + y \equiv a \,(\text{mod } n)$, the reductions mod n of (x, y) must be one of $(\overline{b}_1, \overline{c}_1)$, ..., $(\overline{b}_k, \overline{c}_k)$, and therefore (x, y) is in one of the sets $B_n(b_1) \times B_n(c_1)$,..., $B_n(b_k) \times B_n(c_k)$. This shows that

$$A^{-1}(B_n(a)) \subseteq \bigcup_{i=1}^{k} \Big(B_n(b_i) \times B_n(c_i) \Big) .$$

Conversely, if $(x, y) \in B_n(b_i) \times B_n(c_i)$ for one of the pairs $(\overline{b}_i, \overline{c}_i)$ above, then $x + y \equiv \overline{b}_i + \overline{c}_i = a$, and so $(x, y) \in A^{-1}(B_n(a))$. This shows the other direction of the inclusion,

$$\bigcup_{i=1}^{k} \Big(B_n(b_i) \times B_n(c_i) \Big) \subseteq A^{-1}(B_n(a)),$$

and therefore that

$$A^{-1}(B_n(a)) = \bigcup_{i=1}^{k} \Big(B_n(b_i) \times B_n(c_i) \Big) .$$

Since $A^{-1}(B_n(a))$ is a union of open sets, it is an open set. Since the sets of the form $B_n(a)$ are a base for the topology on $\mathbb{Z}$, this shows that A is continuous.

<u>Solution using the diagram</u>. Let us fix a positive integer n. Since reduction mod n is a ring homomorphism, the diagram below commutes, where the vertical maps are reduction mod n, and the horizontal maps are addition :

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \xrightarrow{\quad S \quad} & \mathbb{Z} \\ {\scriptstyle r_2}\downarrow & & \downarrow{\scriptstyle r_1} \\ (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\ s\ } & \mathbb{Z}/n\mathbb{Z} \end{array}$$

Instead of the 'A' used for the addition map in the problem, here we are using S (for "sum") for the map from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, and s (again for "sum") for the sum map mod n.

Now also fix $a \in \mathbb{Z}$ and let $\overline{a}$ denote the image of a in $\mathbb{Z}/n\mathbb{Z}$.

Since $B_n(a) = r_1^{-1}(\{\overline{a}\})$, by commutativity of the diagram, and functoriality of pullback we have

$$S^{-1}(B_n(a)) = S^{-1}(r_1^{-1}(\{\overline{a}\}) = (r_1 \circ S)^{-1}(\{\overline{a}\}) = (s \circ r_2)^{-1}(\{\overline{a}\}) = r_2^{-1}(s^{-1}\{\overline{a}\}).$$

But, $s^{-1}(\{\overline{a}\})$ is a subset T of $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ and hence a finite set, of the form

$$T = \{(\overline{b}_1, \overline{c}_1),\ (\overline{b}_2, \overline{c}_2),\ \ldots (\overline{b}_k, \overline{c}_k)\}$$

for some elements in $(\overline{b}_i,\ \overline{c}_i)$ in $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$. For each of these, $r_2^{-1}(\{(\overline{b}_i, \overline{c}_i)\}) = B_n(b_i) \times B_n(c_i) \subseteq \mathbb{Z} \times \mathbb{Z}$. Therefore

$$S^{-1}(B_n(a)) = r_2^{-1}(T) = \bigcup_{i=1}^{k} \Big(B_n(b_i) \times B_n(c_i)\Big).$$

Since $S^{-1}(B_n(a))$ is a union of open sets, it is an open set.

This argument holds for any n and any a. Therefore, for each open set of the form $B_n(a)$, $S^{-1}(B_n(a))$ is open. Since these sets form a base for the topology on $\mathbb{Z}$, S is a continuous map.

(d) <u>"Down to earth" solution</u>. As in the solution for addition, let us fix $n \geqslant 1$ and an $a \in \mathbb{Z}$. We want to show that $M^{-1}(B_n(a))$ is an open set. A point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is $M^{-1}(B_n(a))$ if and only if $xy \equiv a \pmod{n}$.

Let $(\overline{b}_1, \overline{c}_1)$, $\ldots$, $(\overline{b}_k, \overline{c}_k)$ be the finitely many solutions to $xy \equiv a \pmod{n}$. Then if $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ and $xy \equiv a \pmod{n}$, the reductions $(\overline{x}, \overline{y})$ mod n must be one of the $(\overline{b}_1, \overline{c}_1)$,$\ldots$, $(\overline{b}_k, \overline{c}_{,})$. As before this shows that

$$M^{-1}(B_n(a)) \subseteq \bigcup_{i=1}^{k} \Big(B_n(b_i) \times B_n(c_i)\Big).$$

The opposite inclusion follows by a similar argument. If $(x, y) \in B_n(b_i) \times B_n(c_i)$ then

$$xy \equiv \overline{b}_1 \overline{c}_i = a \,(\mathrm{mod}\ n).$$

Therefore,

$$M^{-1}(B_n(a)) = \bigcup_{i=1}^{k} \Big(B_n(b_i) \times B_n(c_i) \Big),$$

and so $M^{-1}(B_n(a))$ is an open set.

Since the sets $\{B_n(a)\}_{n \geqslant 1, a \in \mathbb{Z}}$ are a base for the topology on $\mathbb{Z}$, this shows that M is a continuous map.

<u>Solution using the diagram</u>. Fix n. Since reduction mod n is a ring homomorphism, the diagram below commutes, where the vertical maps are reduction mod n, and the horizontal maps multiplication.

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \xrightarrow{\quad M \quad} & \mathbb{Z} \\ {\scriptstyle r_2} \big\downarrow & & \big\downarrow {\scriptstyle r_1} \\ (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\ m\ } & \mathbb{Z}/n\mathbb{Z} \end{array}$$

As before, commutativity of the diagram implies that for any $a \in \mathbb{Z}$,

$$M^{-1}(B_n(a)) = r_2^{-1}(m^{-1}\{\overline{a}\}).$$

As before, $m^{-1}(\{\overline{a}\})$ is some finite set,

$$T = \{(\overline{b}_1, \overline{c}_1),\, (\overline{b}_2, \overline{c}_2),\, \ldots (\overline{b}_k, \overline{c}_k)\},$$

and so

$$M^{-1}(B_n(a)) = r_2^{-1}(T) = \bigcup_{i=1}^{k} \Big(B_n(b_i) \times B_n(c_i) \Big),$$

and therefore $M^{-1}(B_n(a))$ is an open set.

Since this is true for each $a \in \mathbb{Z}$, and each $n \geqslant 1$, and since sets of the form $B_n(a)$ form a base for the topology, multiplication is continuous in this topology.

2.

(a) Let

$$Z_1 = \Big\{ (x, y) \in \mathbb{R}^2 \,\Big|\, x \geqslant y \Big\} \quad \text{and} \quad Z_2 = \Big\{ (x, y) \in \mathbb{R}^2 \,\Big|\, x \leqslant y \Big\}.$$

Both Z_1 and Z_2 are closed subsets of $\mathbb{R}^2$. For instance, if we consider the continuous function $f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x, y) = x - y$ then $Z_1 = f^{-1}([0, \infty))$ and $Z_2 = f^{-1}((-\infty, 0])$. We also have that $Z_1 \cup Z_2 = \mathbb{R}^2$.

$\max|_{Z_1}$ is the function $(x, y) \mapsto x$ and $\max|_{Z_2}$ is the function $(x, y) \mapsto y$, and both these functions are continuous (for instance, they are equal to the projection functions p_1 and p_2 respectively). Therefore, by our proposition from class, max is a continuous function on $\mathbb{R}^2$.

(b) If f and g are continuous functions from X to $\mathbb{R}$, then by the universal property of the product the function $u(x) = (f(x), g(x))$ is a continuous function from X to $\mathbb{R}^2$. By part (a), the function max is continuous on $\mathbb{R}^2$, therefore the composite function $\max \circ u$, i.e., the function

$$(\max \circ u)(x) = \max(u(x)) = \max(f(x), g(x))$$

is a continuous function on X.

Note : It is also easy to argue directly that $\max(f(x), g(x))$ is a continuous function. For any open interval $(a, b) \subseteq \mathbb{R}^2$,

$$\max(f,g)^{-1}((a,b)) = \left(f^{-1}((a,b)) \cap g^{-1}((-\infty,b))\right) \bigcup \left(f^{-1}((-\infty,b)) \cap g^{-1}((a,b))\right).$$

Since f and g are continuous, the set on the right hand side of the equation above is open. But, it is still good to remember this strategy of factoring a map into simpler maps. It is a general strategy for "breaking down" a problem about a function into simpler steps.

3. One way to make such a homotopy is to follow the picture below. To do that, for $t \in [0, 1]$ set

$$a(t) = \tfrac{1}{3} - \tfrac{t}{12} = \tfrac{4-t}{12} \quad \text{and} \quad b(t) = \tfrac{2}{3} - \tfrac{t}{6} = \tfrac{4-t}{6}.$$

These are the lines on the edges of the γ_a, γ_b and γ_c regions of the homotopy. Then we can write the homotopy $H\colon [0, 1] \times [0, 1] \longrightarrow X$ as :

$$H(s,t) = \begin{cases} \gamma_a\left(\frac{s}{a(t)}\right) & \text{for } s \in [0, a(t)] \\ \gamma_b\left(\frac{s-a(t)}{b(t)-a(t)}\right) & \text{for } s \in [a(t), b(t)] \\ \gamma_c\left(\frac{s-b(t)}{1-b(t)}\right) & \text{for } s \in [b(t), 1]. \end{cases}$$