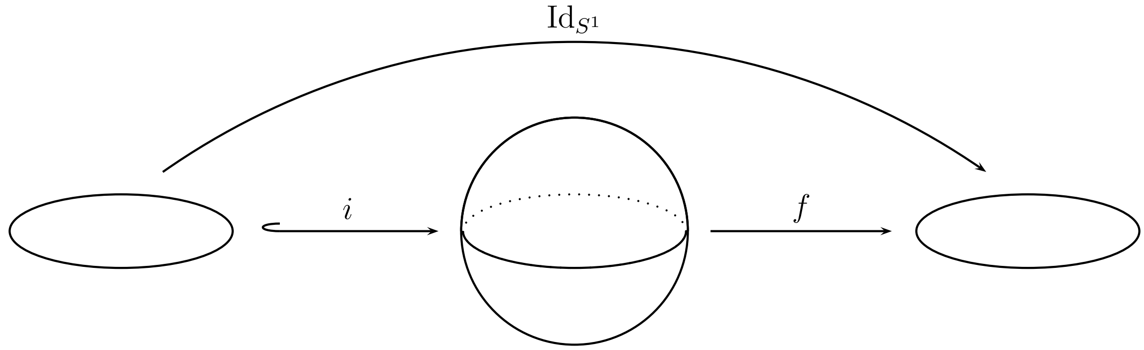
1.

(a) Given $A \subseteq X$ (with X a topological space) a retract of X onto A, or just a retract onto A, is a continuous map $f\colon X \longrightarrow A$ such that $f|_A = \mathrm{Id}_A$, or equivalently such that $f \circ i_A = \mathrm{Id}_A$, where $i_A\colon A \hookrightarrow X$ is the inclusion map.

(b) As in the proof from class, let $i\colon S^1 \hookrightarrow S^2$ be the inclusion of S^1 into S^2, and assume that there is a retract $f\colon S^2 \longrightarrow S^1$.



Let us pick the common basepoint $s_0 = (1, 0, 0)$ for S^1 and S^2. Applying the functor π_1 converts the maps of pointed topological spaces above into maps of groups :

$$\mathbb{Z} \xrightarrow{\quad i_* \quad} 0 \xrightarrow{\quad f_* \quad} \mathbb{Z}.$$

Moreover, since π is a functor,

$$f_* \circ i_* \overset{\text{(i)}}{=\!=} (f \circ i)_* \overset{\text{(ii)}}{=\!=} (\mathrm{Id}_{S_1})_* \overset{\text{(iii)}}{=\!=} \mathrm{Id}_{\mathbb{Z}},$$

with reasons

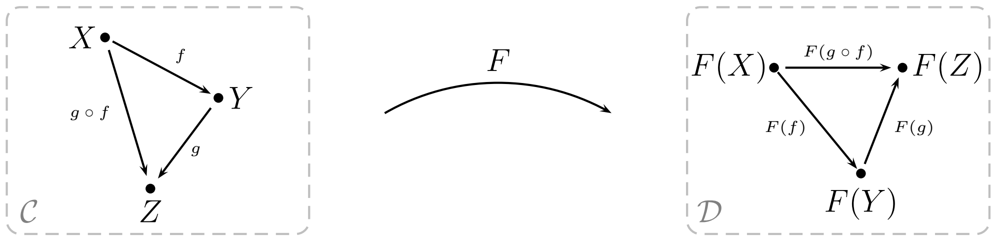
(*i*) Functors are compatible with composition;

(*ii*) $f \circ i = \mathrm{Id}_{S^1}$ (that is our assumption!); and

(*iii*) Functors take the identity map to the identity map.

Since the composition $f_* \circ i_*$ is clearly the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ which sends everything to zero, $f_* \circ i_* \neq \mathrm{Id}_{\mathbb{Z}}$. This contradiction shows that such a map f does not exist.

(c) Given two categories $\mathcal{C}$ and $\mathcal{D}$, a (covariant) functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is a map which assigns to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$, and for each morphism $f\colon X \longrightarrow Y$ between objects of $\mathcal{C}$ gives a morphism $F(f)\colon F(X) \longrightarrow F(Y)$ in $\mathcal{D}$.



The assigment of morphisms is required to respect composition. I.e., for objects X, Y, and Z in $\mathcal{C}$, and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ we must have

$$F(g \circ f) = F(g) \circ F(f) \text{ in } \mathrm{Hom}_{\mathcal{D}}(F(X), F(Z)).$$

We also require that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for all $X \in \mathcal{C}$.

(d) Every part of the definition of functor was used in the argument in (b). First, we need that the functor takes each pointed topological space to a group, i.e., takes (S^1, s_0) to $\mathbb{Z}$ and (S^2, s_0) to 0. Second, we need that the functor takes maps between these topological spaces to maps between the corresponding groups. For instance, the maps i and f are taken to group homomorphisms between the corresponding fundamental groups.

Next, we used the fact that functors are compatible with composition. That is exactly reason (i) in the argument in (b). Finally, we need to use the fact that a functor takes the identity morphism to the identity morphism, in this case, the identity map on S^1 to the identity map on $\mathbb{Z}$. That was reason (iii) in the argument in (b).

2.

(a) If $x^2 + y^2 + z^2 = 1$, the vector

$$(1-t)(x,y,z) + t(0,0,1) = ((1-t)x,\ (1-t)y,\ (1-t)z + t)$$

has squared length

$$((1-t)x)^2 + ((1-t)y)^2 + ((1-t)z_t)^2 = (1-t)^2x^2 + (1-t)^2y^2 + (1-t)^2z^2 + 2t(1-t)z + t^2$$
$$= (1-t)^2\left(x^2+y^2+z^2\right) + 2t(1-t)z + t^2 = (1-t)^2 \cdot 1 + 2t(1-t)z + t^2.$$

Thus the vector has length $\sqrt{(1-t)^2 + 2t(1-t)z + t^2}$.

For use in part (b) and part (f) below, note that as long as $(x, y, z) \neq (0, 0, -1)$ (and $x^2 + y^2 + z^2 = 1$) this length can never be zero.

Geometrically the reason is that the path $(1-t)(x, y, z) + t(0, 0, 1)$ for $t \in [0, 1]$ is the segment connecting the point (x, y, z) and $(0, 0, 1)$ on the sphere, and this only goes through $(0, 0, 0)$ if the starting point is $(x, y, z) = (0, 0, -1)$.

To see this algebraically, start by writing the squared length as a polynomial in t :

$$2(1-z)t^2 + 2(z-1)t + 1.$$

On the sphere we have $-1 \leqslant z \leqslant 1$. The discriminant of this polynomial is

$$(2(z-1))^2 - 4(2(1-z) \cdot 1)) = 4(z+1)(z-1).$$

So, if $-1 < z < 1$ then the discriminant is negative, and hence the polynomial has no roots (i.e., the length is never zero). If $z = 1$ the polynomial becomes the constant polynomial 1. It is only when $z = -1$ (and hence when the point is $(0, 0, -1)$) that the polynomial has a root, occuring when $t = \frac{1}{2}$, i.e., when the vector is $(0, 0, 0)$.

(b) By part (a) the denominator of

$$\frac{\big((1-t)x,\ (1-t)y,\ (1-t)z + t\big)}{\sqrt{(1-t)^2 + 2(1-t)tz + t^2}}$$

is the length of the numerator, and by the discussion above the denominator is never zero (so this expression makes sense). Therefore, for each $(x, y, z) \in S^2$ and $t \in [0, 1]$ the expression above is a vector of length 1, and so on S^2.

To see that the image of F lies in U we need to check that the output of F can never be $(0, 0, -1)$. When $t = 1$, $F((x, y, z), 1) = (0, 0, 1)$, no matter what (x, y, z) is. If x or y is $\neq 0$, then for $t \neq 1$ $(1-t)x$ or $(1-t)y$ is also $\neq 0$ and so the output vector can never be equal to $(0, 0, -1)$. Thus, the only possibility is to have $x = 0$ and $y = 0$, in which case $z = 1$ or $z = -1$. In the first case $F((0, 0, 1), t) = (0, 0, 1)$ for all values of t. Only in the second case, $F((0, 0, -1), t)$ can the output of F be equal to $(0, 0, -1)$, and then only when $t = 0$.

Since $(0, 0, -1)$ is not in U, we conclude that for all $(x, y, z) \in U$, $F((x, y, z), t) \in U$.

(c) When $t = 0$, $F((x, y, z), 0) = (x, y, z)$, so f_0 is the map Id_U.

(d) When $t = 1$, $F((x, y, z), 1) = (0, 0, 1)$, so f_1 is the function sending all of U to $(0, 0, 1)$.

(e) The map H is the composition

$$[0,1]\times[0,1] \xrightarrow{\gamma\times \mathrm{Id}_{[0,1]}} X\times[0,1] \xrightarrow{F} X.$$

Since F is the composition of continuous maps, H is continuous.

Since γ is a loop based at $(0,0,1)$, $\gamma(0)=(0,0,1)=\gamma(1)$. Restricted to the edge $\{0\}\times[0,1]$ we therefore have

$$H(0,t)=F(\gamma(0),t))=\frac{(1-t)(0,0,1)+t(0,0,1)}{\sqrt{(1-t)^2+2t(1-t)+t^2}}=\frac{(0,0,1)}{\sqrt{(1-t)+t)^2}}=\frac{(0,0,1)}{1}=(0,0,1).$$

By essentially the same calculation, restricting H to the edge $1\times[0,1]$ we have

$$H(1,t)=F(\gamma(1),t))=\frac{(1-t)(0,0,1)+t(0,0,1)}{\sqrt{(1-t)^2+2t(1-t)+t^2}}=(0,0,1).$$

On the edge $[0,1]\times\{0\}$ we have $H(s,0)=F(\gamma(s),0)=\gamma(s)$, since by (c) $F((x,y,z),0)=(x,y,z)$ for all $(x,y,z)\in U$.

On the edge $[0,1]\times\{1\}$ we have $H(s,1)=F(\gamma(s),1)=(0,0,1)$, since by (d) $F((x,y,z),1)=(0,0,1)$, no matter what $(x,y,z)\in U$ is.

Therefore H is a loop homotopy between $\gamma(s)$ and the constant loop based at $(0,0,1)$.

(f) This calculation does not work if we allow $(0,0,-1)$ into the domain. The issue is the one discussed after part (a). If we start at $(0,0,-1)$, then when $t=\frac{1}{2}$ the path $(1-t)(0,0,-1)+t(0,0,1)$ is at the origin and has length zero. Therefore we cannot divide by the length to get a vector on the unit sphere.

Fixing $(x,y,z)\in U$, $F((x,y,z),t)$ is the "flow upwards to $(0,0,1)$", as shown in the picture below.