1.

(a) If there were a point $x \in A_1 \cap B_1$ then we would have $d(x, A_1) = 0$ and $d(x, B_1) = 0$, and hence the first entry of $f(x)$ would be $0/0$, which does not make sense. Similarly if there were a point $x \in A_2 \cap B_2$ the second entry of $f(x)$ would again by $0/0$.

On the other hand, since A_1 and B_1 are closed (A_1 is closed by hypothesis, and B_1 is the image of A_1 under the homeomorphism $x \mapsto \overline{x}$), $d(x, A_1) = 0$ only if $x \in A_1$, and $d(x, B_1) = 0$ only if $x \in B_1$. Therefore the condition $A_1 \cap B_1 = \varnothing$ is sufficent to ensure that the first coordinate of f makes sense as a function. Similarly, the condition $A_1 \cap B_1 = \varnothing$ ensures that the second coordinate of f is a well-defined function on S^2.

(b) The key to the statement is that for any point $x \in S^2$, $d(\overline{x}, A_1) = d(x, B_1)$, and similarly for A_2 and B_2. To see this, we note that $x \mapsto \overline{x}$ is a bijection of the sphere with itself and that this map preserves distance in $\mathbb{R}^2$. I.e., for any x, $y \in S^2$, $d(x, y) = d(\overline{x}, \overline{y})$. Therefore

$$d(\overline{x}, A_1) = \inf\{d(\overline{x}, z) \mid z \in A_1\} = \inf\{d(x, \overline{z}) \mid z \in A_1)\} = \inf\{d(x, \overline{z}) \mid \overline{z} \in B_1\} = d(x, B_1).$$

Using this formula, we can write f as

$$f(x) = \left(\frac{d(x, A_1)}{d(x, A_1) + d(\overline{x}, A_1)}, \frac{d(x, A_2)}{d(x, A_2) + d(\overline{x}, A_2)}\right).$$

Using this version, and the fact that $\overline{\overline{x}} = x$, we compute that

$$\begin{aligned} f(x) + f(\overline{x}) &= \left(\tfrac{d(x,A_1)}{d(x,A_1)+d(\overline{x},A_1)}, \tfrac{d(x,A_2)}{d(x,A_2)+d(\overline{x},A_2)}\right) + \left(\tfrac{d(\overline{x},A_1)}{d(x,A_1)+d(\overline{x},A_1)}, \tfrac{d(\overline{x},A_2)}{d(x,A_2)+d(\overline{x},A_2)}\right) \\ &= \left(\tfrac{d(x,A_1)+d(\overline{x},A_1)}{d(x,A_1)+d(\overline{x},A_1)}, \tfrac{d(x,A_2)+d(\overline{x},A_2)}{d(x,A_2)+d(\overline{x},A_2)}\right) = (1, 1). \end{aligned}$$

(c) Since $f(x) = f(\overline{x})$, and $f(x) + f(\overline{x}) = (1, 1)$ we conclude that $f(x) = (\frac{1}{2}, \frac{1}{2})$ and $f(\overline{x}) = (\frac{1}{2}, \frac{1}{2})$.

Since neither coordinate of $f(x)$ is zero, $x \notin A_1$ and $x \notin A_2$.

(d) Similarly, since neither coordinate of $f(\overline{x})$ is zero, $\overline{x} \notin A_1$ and $\overline{x} \notin A_2$.

(e) By hypothesis, $A_1 \cup A_2 \cup A_3 = S^2$ Therefore (c) implies that $x \in A_3$ and (d) implies that $\overline{x} \in A_3$.

2.

(a) Since $g(z) = f(z)/z$, and $s_0 = 1 \in \mathbb{C}$, $g(s_0) = f(s_0)/s_0 = f(1)/1 = f(1) = s_1$.

(b) The composition $g \circ \gamma$ is

$$(g \circ \gamma)(s) = g(\gamma(s)) = \frac{f(\gamma(s))}{\gamma(s)}.$$

By hypothesis $p(\tilde{\gamma}(s)) = e^{2\pi i \tilde{\gamma}(s)} = f(\gamma(s))$ for all $s \in [0,1]$. Therefore

$$p(\tilde{\gamma}(s) - s) = e^{2\pi i(\tilde{\gamma}(s) - s)} = \frac{e^{2\pi i \tilde{\gamma}(s)}}{e^{2\pi i s}} = \frac{f(\gamma(s))}{\gamma(s)} = (g \circ \gamma)(s)$$

for all $s \in [0,1]$. Thus the function $s \mapsto \tilde{\gamma}(s) - s$ is a lift of $g \circ \gamma$.

(c) Using this lift we compute that

$$\deg(g) = (\tilde{\gamma}(1) - 1) - (\tilde{\gamma}(0) - 0) = (x_1 + m - 1) - (x_1 - 0) = m - 1.$$

(d) The set U is homeomorphic to the interval $(0,1)$. For instance the covering map p gives a continuous bijection of $(0,1)$ with U, and the continuous function $\log(z)/2\pi i$, defined on $\mathbb{C} \backslash \mathbb{R}_{\geqslant 0}$, (and correctly normalized), gives a continuous bijection in the opposite direction.

Homeomorphic topological spaces have the same fundamental group (this follows from the functoriality of π_1). The interval $(0,1)$ is convex, so has trivial fundamental group. Therefore $\pi_1(U, s_1) = 0$.

(e) If $1 \notin \mathrm{Im}(g)$ then we have the factorization

(2.1)

$$\begin{tikzcd}
(S^1, s_0) \arrow[r] \arrow[rr, bend left, "g"] & (U, s_1) \arrow[r, "i"] & (S^1, s_1).
\end{tikzcd}$$

Hence, applying the functor π_1, we get

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[rr, bend left, "g_*"] & 0 \arrow[r, "i_*"] & \mathbb{Z}.
\end{tikzcd}$$

Thus the map g_* is the zero map. Since the map is also multiplication by $m-1$, we conclude that $m-1=0$.

In summary, the implications are

f has no fixed point $\implies 1 \notin \mathrm{Im}(g) \implies g$ factors as in (**??**) $\implies m-1=0$.

Thus, if $m \neq 1$, f has a fixed point.