DUE DATE: SEPT. 13, 2024

Below are some examples of topologies on $\mathbb{R}$ (most of which we have already seen), along with names, not all official, to make it easier to refer to them on this assignment.

STD : The standard topology.

FC : The finite complement topology (a nonempty subset $U \subseteq \mathbb{R}$ is open iff its complement $\complement_{\mathbb{R}} U = \mathbb{R} \setminus U = \mathbb{R} - U$ is finite).

CC : The countable complement topology (a nonempty subset U $\subseteq \mathbb{R}$ is open iff its complement is countable).

ARR : The open sets are $\varnothing$, $\mathbb{R}$, and intervals of the form (a, ∞), with $a \in \mathbb{R}$.

TRIV : The trivial topology : $\{\varnothing, \mathbb{R}\}$.

DISC : The discrete topology (all subsets of $\mathbb{R}$ are open).

Note that "countable" means "finite or countably infinite", and that the empty set is (of course) an open set in the topologies FC and CC.

1. The topologies above are partially ordered by inclusion (or, to use the terminology from class, by which topology is finer than another). Draw the Hasse diagram corresponding to these inclusions. (E.g., draw the diagram where a vertical line connecting the topologies means that the lower topology is contained in the higher topology, like the diagram we have drawn for the four possible topologies on the set $\{a, b\}$.)

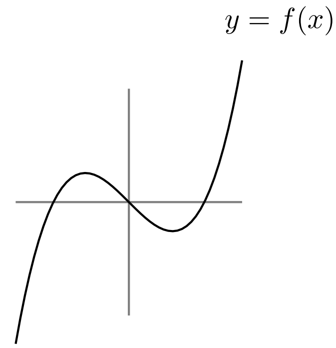
NOTE: If $\tau_1 \subseteq \tau_2$, it follows that the closed subsets of the topology τ_1 are a subset of the closed subsets of the topology τ_2 (i.e., the inclusion of the closed sets goes in the same direction). This may help when deciding some of the inclusions in this question.

2.

(a) Suppose that y_1, $y_2 \in \mathbb{R}$, with $y_1 \neq y_2$. Explain why (in the standard topology on $\mathbb{R}$) there are open sets V_1, $V_2 \subseteq \mathbb{R}$ with $y_1 \in V_1$, $y_2 \in V_2$ and $V_1 \cap V_2 = \varnothing$.

(b) Suppose that X is a topological space with the property that the intersection of any two nonempty open sets is again nonempty. (I.e., if U_1 and U_2 are open in X, and both are nonempty, then $U_1 \cap U_2 \neq \varnothing$.) Prove that all continuous functions $f\colon X \longrightarrow \mathbb{R}$ (where $\mathbb{R}$ has the standard topology) are constant.

(c) Which of the topologies on $\mathbb{R}$ listed above satisfy the condition in (b)?



3. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the function $f(x) = x^3 - x$. The graph of f is shown at right. In this question we will investigate whether f is continuous for various combinations of topologies on the source $\mathbb{R}$ and on the target $\mathbb{R}$.

topology on the target

topology on the source

	STD	FC	CC	ARR	TRIV	DISC
STD	(a)			(b)		
FC		(e)	(f)			
CC						
ARR	(c)			(d)		
TRIV					(g)	(h)
DISC					(i)	

In each of the cases labelled above, decide whether f is continuous, and explain your reasoning.

Reminder : For f to be continuous, it is equivalent that the inverse images of closed sets be closed.

4. Let $f\colon X \longrightarrow Y$ be a map of sets, and let τ_Y be a topology on Y. Define $\tau_X \subseteq \mathcal{P}(X)$ by

$$\tau_X = \left\{ f^{-1}(V) \,\middle|\, V \in \tau_Y \right\}.$$

I.e., $U \in \tau_X$ if and only if there is a $V \in \tau_Y$ with $f^{-1}(V) = U$.

(a) Prove that τ_X is a topology on X.

By virtue of the definition of τ_X, if we use τ_X for the topology on X, then f is continuous.

(b) Show that τ_X is the coarsest topology which makes f continuous (with respect to the fixed topology τ_Y on Y). I.e., if τ'_X is any topology on X such that f is continuous with respect to τ'_X and τ_Y then $\tau_X \subseteq \tau'_X$.

Now let us try a similar construction in the other direction. Fix a topology τ_X on X, and define $\tau_Y \subseteq \mathcal{P}(Y)$ by

$$\tau_Y = \left\{ V \subseteq Y \,\middle|\, f^{-1}(V) \in \tau_X \right\}.$$

I.e., $V \in \tau_Y$ if and only if $f^{-1}(V) \in \tau_X$.

(c) Prove that τ_Y is a topology on Y.

(d) Show that τ_Y is the finest topology on Y which makes f continuous (with respect to the fixed topology τ_X on X). I.e., if τ'_Y is any topology on Y which makes f continuous with respect to τ_X and τ'_Y then $\tau'_Y \subseteq \tau_Y$.