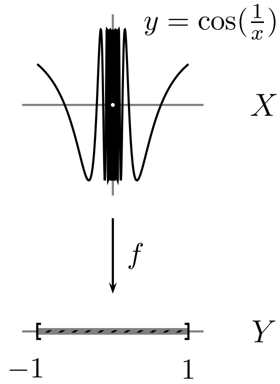
Math 347 — Homework Assignment 2

DUE DATE: SEPT. 20, 2024



1. Let

$$X = \left\{ \left(x, \cos\left(\tfrac{1}{x}\right)\right) \;\middle|\; x \in [-1,0) \cup (0,1] \right\} \cup \{(0,0)\},$$
$$Y = [-1,1]$$

and let $f \colon X \longrightarrow Y$ be projection onto the first coordinate, i.e., $f(x,y) = x$ for all $(x,y) \in X$.

The map f is a bijection. The inverse map is $g \colon Y \longrightarrow X$ given by

$$g(x) = \begin{cases} (x,\, \cos(\frac{1}{x})) & \text{if } x \neq 0 \\ (0,0) & \text{if } x = 0 \end{cases}$$

Let X and Y have the subspace topologies, induced from $X \subseteq \mathbb{R}^2$ and $Y \subseteq \mathbb{R}$ respectively, where $\mathbb{R}^2$ and $\mathbb{R}$ are given the standard topologies.

In this problem we will show that f is continuous, but g is not continuous. I.e., that even though f is a continuous bijection, f is not a homeomorphism.

(a) Let $p \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the map $p(x,y) = x$ (i.e., projection onto the first coordinate). Briefly explain why p is continuous.

(b) Explain why $p \circ i_X$ is continuous, where $i_X \colon X \hookrightarrow \mathbb{R}^2$ is the inclusion map.

(c) Prove that f is continuous.

SUGGESTION : Now might be a good time to use part (c) of the theorem on properties and characterization of the subspace topology. Note that the way that f and g are used in that theorem does not match the way that f and g are used in this problem, so don't confuse them!

Now we will show that g is not continuous.

(d) Explain why (to show that g is not continuous) it is enough to find a closed set $W \subseteq X$ such that $f(W)$ is not closed in Y.

(e) Show that the set $W = \left\{ \left(\frac{1}{2\pi n}, 1\right) \;\middle|\; n \in \mathbb{Z},\, n \neq 0 \right\}$ is a closed set in X.

SUGGESTION : Maybe W is the intersection of X with a set which is closed in $\mathbb{R}^2$ (which is how, with the subspace topology, all closed sets arise).

(f) Prove that g is not continuous.

2. In this problem we will establish some of the formal properties of the closure operator. Let (X, τ_X) be a topological space, and $A \subseteq X$ a subset of X. We define $\overline{A}$, the *closure of* A, by

$$\overline{A} = \bigcap_{\substack{A \subseteq Z \\ Z \subseteq X \text{ closed}}} Z.$$

I.e., $\overline{A}$ is defined to be the intersection of all closed sets Z which contain A.

(a) Explain (or show) why $\overline{A}$ has the following three properties.

- (a1) $\overline{A}$ is closed (as its name suggests!).
- (a2) $A \subseteq \overline{A}$.
- (a3) If W is any closed set with $A \subseteq W$, then $\overline{A} \subseteq W$.

NOTE : These three properties all follow directly from the definition above (and, at one point the axioms for closed sets), and so that is the best way to argue.

(b) Show that these three properties characterize $\overline{A}$. I.e., if Z is a set which

- (b1) is closed
- (b2) contains A, and has the property that
- (b3) if W is any closed set with $A \subseteq W$, then $Z \subseteq W$,

then $Z = \overline{A}$.

SUGGESTION : Use the conditions to show that $\overline{A} \subseteq Z$ and that $Z \subseteq \overline{A}$.

(c) If $V \subseteq X$ is a closed set, show that $\overline{V} = V$.

SUGGESTION : Consider (*a*2) and (*a*3)!

(d) For any subset $A \subseteq X$, show that $\overline{\overline{A}} = \overline{A}$.

Let A, $B \subset X$ be any two subsets of X.

(e) Explain why $\overline{A} \cup \overline{B}$ is closed.

(f) Explain why $A \cup B \subseteq \overline{A} \cup \overline{B}$.

(g) Show that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

(h) Explain why $A \subseteq \overline{A \cup B}$.

(i) Show that $\overline{A} \subseteq \overline{A \cup B}$. (And so similarly $\overline{B} \subseteq \overline{A \cup B}$.)

(j) Show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

(k) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

NOTE : From (k) (and induction) it follows that the closure operator commutes with any finite union.