1. Let $\mathcal{B} = \left\{ [a,b) \,\middle|\, a,\, b \in \mathbb{R},\, a < b \right\} \subset \mathcal{P}(\mathbb{R})$.

(a) Show that the elements of $\mathcal{B}$ form a base for the topology they generate.

Let $\tau_{\mathrm{S}}$ be the topology on $\mathbb{R}$ generated by $\mathcal{B}$. This topology is known as the "right half-open topology", and $\mathbb{R}$ with this topology is also known as the "Sorgenfrey line".

(b) Show that each interval $(a,b)$ $(a,\, b \in \mathbb{R},\, a < b)$ is open in $\tau_{\mathrm{S}}$.

(c) Show that each element of $\mathcal{B}$ is closed in $\tau_{\mathrm{S}}$.

From (b), $\tau_{\mathrm{S}}$ is finer than the standard topology on $\mathbb{R}$. Thus, in the diagram from **H1 Q1** we would put $\tau_{\mathrm{S}}$ between the standard topology and the discrete topology. We have never seen a topology strictly between these two before. Perhaps there isn't one. I.e., perhaps this topology is really either the standard topology or the discrete topology.

From (c) we see that $\tau_{\mathrm{S}}$ can be generated by sets which are both open and closed in $\tau_{\mathrm{S}}$. Perhaps that means every open set in $\tau_{\mathrm{S}}$ is also closed. (A statement which *is* true for the discrete topology.)

Let us now see that both of these guesses are incorrect.

(d) Let $x_0$ be any point of $\mathbb{R}$. Show that the set $\{x_0\}$ is closed in $\tau_{\mathrm{S}}$.

(e) Show that the set $\{x_0\}$ is not open in $\tau_{\mathrm{S}}$.

(f) Show that $\tau_{\mathrm{S}}$ is not the discrete topology,

(g) ...and show that not every open set in $\tau_{\mathrm{S}}$ is closed.

Of course, something similar occurs if we start with the set $\mathcal{B}' = \{(a,b] \mid a,\, b \in \mathbb{R},\, a < b\}$ as a generating set for a topology. Encouraged by this, we may think : What about taking the topology generated by $\mathcal{B}'' = \{[a,b] \mid a,\, b \in \mathbb{R},\, a < b\}$?

(h) Show that the topology generated by $\mathcal{B}''$ is the discrete topology on $\mathbb{R}$.

2. Let $\tau_X$ be a topology on a set $X$. In **H2 Q2** we explored the closure operator on subsets of $X$. The closure operator has the following properties :

(cl1) $\overline{\varnothing} = \varnothing$

(cl2) For all $A \subseteq X$, $A \subseteq \overline{A}$.

(cl3) For all $A \subseteq X$, $\overline{\overline{A}} = \overline{A}$

(cl4) For all $A$, $B \subseteq X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

[(cl1) is clear from the definition in **H2 Q2**, while (cl2)–(cl4) are **H2 Q2** (a2), (d), and (k) respectively.]

In developing the theory of topological spaces we have started from the axioms for open sets. One could equally well start from the axioms for closed sets. Kuratowski, an early researcher in the ideas and foundations of topology, explored the possibility that having an operator satisfying the conditions above was equivalent to giving a topology.

Let $X$ be a set, and $c\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be a function satisfying the *Kuratowski closure axioms* :

(K1) $c(\varnothing) = \varnothing$

(K2) For all $A \subseteq X$, $A \subseteq c(A)$.

(K3) For all $A \subseteq X$, $c(c(A)) = c(A)$

(K4) For all $A$, $B \subseteq X$, $c(A \cup B) = c(A) \cup c(B)$.

Let us call a subset $Z \subseteq X$ *closed* if $c(Z) = Z$. In this problem we will show that these sets satisfy the axioms to be the closed sets of a topology, and that $c$ is the closure operator in that topology. Using only the properties above :

(a) Show that $\varnothing$ is a closed set.

(b) Show that $X$ is a closed set.

(c) Show that the finite union of closed sets is a closed set.

Let $A$, $B \subseteq X$, with $A \subseteq B$, and set $D = B \setminus A$ (i.e, $D = \complement_B A$), so that $B = A \cup D$.

(d) Show that $c(A) \subseteq c(B)$

Now let $Z_i$, $i \in I$, be an arbitrary family of closed sets, and set $W = \bigcap_{i \in I} Z_i$.

(e) Explain why $W \subseteq Z_i$ for each $i$.

(f) Show that $c(W) \subseteq Z_i$ for each $i$.

(g) Show that $c(W) \subseteq W$.

(h) Show that $c(W) = W$ (i.e., show that the arbitrary intersection of closed sets is closed).

Thus, the sets $Z \subseteq X$ satsifying $c(Z) = Z$ form the closed sets of a topology $\tau_X$ on $X$. This topology therefore gives us a closure operator $A \mapsto \overline{A}$ (as in **H2 Q2**).

(i) Show that this closure operator agrees with $c$. I.e., show that for any $A \subseteq X$, $\overline{A} = c(A)$.