1. In this problem we will review the general definition of monomorphisms and epimorphisms in a category, and see that in **Set** they are the same as injectivity and surjectivity of maps.

Monomorphisms. A morphism $f\colon X \longrightarrow Y$ in a category $\mathcal{C}$ is called a monomorphism if, for all $Z \in \mathcal{C}$, and maps g_1, $g_2\colon Z \longrightarrow X$ in $\mathcal{C}$, if $f \circ g_1 = f \circ g_2$ then $g_1 = g_2$. (I.e., the only way to have $f \circ g_1 = f \circ g_2$ is if $g_1 = g_2$). Here is a diagram with these maps.

$$\begin{tikzcd} Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Suppose that X and Y are sets, and $f\colon X \longrightarrow Y$ is a map of sets.

(a) If f is injective, prove that f is a monomorphism, i.e., prove that f satisfies the condition above.

(b) Conversely, suppose that f is a monomorphism in **Set**. Show that f is injective

SUGGESTION : One way to prove (b) is to establish the contrapositive : If f is not an injection, then f is not a monomorphism. To follow that approach, assume that f is not an injection, and show how to find a set Z, and maps g_1, $g_2\colon Z \longrightarrow X$, with $g_1 \neq g_2$, such that $f \circ g_1 = f \circ g_2$.

Epimorphisms. A morphism $f\colon X \longrightarrow Y$ in a category $\mathcal{C}$ is called a epimorphism if, for all $Z \in \mathcal{C}$, and maps g_1, $g_2\colon Y \longrightarrow Z$ in $\mathcal{C}$, if $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$. (I.e., the only way to have $g_1 \circ f = g_2 \circ f$ is if $g_1 = g_2$.)

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z. \end{tikzcd}$$

As above, let $f\colon X \longrightarrow Y$ be a map of sets.

(c) If f is surjective, prove that f is an epimorphism, i.e., prove that f satisfies the condition above.

(d) Conversely, assume that f is an epimorphism in **Set**. Prove that f is surjective.

SUGGESTION : As in (b), to show (d) it is probably easier to prove the contrapositive statement.

2. In class we have seen that products are "unique up to unique isomorphism". In this question we will see an example to help understand what that means. Let X_1, X_2, and X_3 be sets. We have the triple product of these sets :

$$X_1 \times X_2 \times X_3 = \Big\{(x_1, x_2, x_3) \,\Big|\, x_1 \in X_1,\ x_2 \in X_2,\ x_3 \in X_3\Big\}.$$

But let us also consider

$$X_1 \times (X_2 \times X_3) = \Big\{(x_1,\ (x_2, x_3)) \,\Big|\, x_1 \in X_1,\ x_2 \in X_2,\ x_3 \in X_3\Big\}.$$

These are not the same sets. For example, the first set is made up of triples, while the second set is made up of pairs (the second element of which also happens to be a pair).

The second set comes with projection maps $q_1\colon X_1 \times (X_2 \times X_3) \longrightarrow X_1$ and $q_2\colon X_1 \times (X_2 \times X_3) \longrightarrow X_2 \times X_3$. But $X_2 \times X_3$ also has its own projection maps $r_1\colon X_2 \times X_3 \longrightarrow X_2$ and $r_2\colon X_2 \times X_3 \longrightarrow X_3$.

Let $p_1 = q_1$, $p_2 = r_1 \circ q_2$, and $p_3 = r_2 \circ q_2$. These are maps from $X_1 \times (X_2 \times X_3)$ to X_1, X_2, and X_3 respectively.

(a) Show that $X_1 \times (X_2 \times X_3)$ with the maps p_1, p_2, and p_3 is a product of X_1, X_2, and X_3 in the category of sets.

Suggestion : Rather than arguing with elements, try and practice using the universal property of products by using the universal properties of $X_2 \times X_3$ and $X_1 \times (X_2 \times X_3)$ to show that $X_1 \times (X_2 \times X_3)$ satisfies the universal property for the product of X_1, X_2, and X_3.

Since $X_1 \times X_2 \times X_3$ is also a product of X_1, X_2, and X_3, by our proposition from class, there is a unique isomorphism $f\colon X_1 \times X_2 \times X_3 \longrightarrow X_1 \times (X_2 \times X_3)$ compatible with their structures as triple products.

(b) What is that map f?

(You do not need to prove that your answer is correct. It is enough to write down what you think f does.)

Note : This type of argument shows that the product construction is associative (at least up to canonical isomorphism), since $X_1 \times X_2 \times X_3 \cong X_1 \times (X_2 \times X_3)$, and similarly $X_1 \times X_2 \times X_3 \cong (X_1 \times X_2) \times X_3$, and so $(X_1 \times X_2) \times X_3 \cong X_1 \times (X_2 \times X_3)$. (Here $\cong$ means "is isomorphic to".) It also shows that if $\mathcal{C}$ is a category where the product of any two elements exist, then products of finitely many elements always exist.