1. In this problem we will consider $\mathbb{R}$ and $\mathbb{R}^2$ with their standard topologies. Define the maps

$$s\colon \mathbb{R}^2 \longrightarrow \mathbb{R} \text{ and } p\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$$

by

$$s(x,y) = x+y \text{ and } p(x,y) = xy.$$

(a) Show that s is continuous.

Suggestion : You can make your solution slightly simpler if you consider the geometry of the map s, and use an answer like that for the solution of **H2 Q1**(a)(2).

(b) Suppose that x_0, y_0, $\epsilon \in \mathbb{R}$, with $0 < \epsilon \leqslant 3$, and that δ_x and δ_y are real numbers satisfying

$$|\delta_x| \leqslant \frac{\epsilon}{3(1+|y_0|)} \text{ and } |\delta_y| \leqslant \frac{\epsilon}{3(1+|x_0|)}.$$

Show that

$$|(x_0+\delta_x)(y_0+\delta_y) - x_0y_0)| \leqslant \epsilon.$$

(c) Show that p is continuous.

Suggestion : In (c) it might be easier to use the base for the topology of $\mathbb{R}^2$ consisting of "open rectangles" (i.e., the standard base for the product topology), as well as, of course, part (b).

Now let X be a topological space, and f, $g\colon X \longrightarrow \mathbb{R}$ continuous maps.

(d) Is the map $X \longrightarrow \mathbb{R}^2$ given by $x \mapsto (f(x), g(x))$ continuous? If so explain how you know.

(e) Prove that $f+g$ and $f \cdot g$ are also both continuous maps (from X to $\mathbb{R}$).

Let $\mathcal{C}(X,\mathbb{R})$ be the set of continuous maps from X to $\mathbb{R}$.

(f) Show that $\mathcal{C}(X,\mathbb{R})$ is a ring. (Here you do not have to give a detailed argument. Instead explain how the steps above show that $\mathcal{C}(X,\mathbb{R})$ is always a ring.)

2. Let $I = \{1, 2, 3, \ldots\}$ be the positive integers, and $\mathbb{R}^\infty = \prod_{i \in I} \mathbb{R}$. I.e., $\mathbb{R}^\infty$ is a product of a countable number of copies of $\mathbb{R}$, and let each copy of $\mathbb{R}$ have the standard topology.

Let $U = (0,1)^\infty$ be the subset of $\mathbb{R}^\infty$ where each coordinate is in the interval $(0,1)$. It might seem that U should be an open subset of $\mathbb{R}^\infty$, and U *is* an open subset in the box topology on $\mathbb{R}^\infty$, but U *is not* an open subset in the product topology on $\mathbb{R}^\infty$.

In this question we will see why, and hopefully also understand a bit better the reasons for the definition of the product topology.

For each $n \in I$ (i.e., each $n \in \{1, 2, 3, \ldots\}$) let $f_n \colon \mathbb{R} \longrightarrow \mathbb{R}$ be the function $f_n(x) = x - \frac{1}{2n}$. Here both source and target $\mathbb{R}$ have the standard topology, and so f_n is a continuous function.

(a) Find $f_n^{-1}((0,1))$.

The functions $\{f_n\}_{n \in I}$ define a unique map $u \colon \mathbb{R} \longrightarrow \mathbb{R}^\infty$, the map which is "$f_n$ in each coordinate", i.e., the map so that $f_n = p_n \circ u$ for each $n \in I$, where p_n is the n-th projection map.

(b) Explain why $u^{-1}(U) = \bigcap_{n \in I} f_n^{-1}((0,1))$.

(c) Find the set in (b).

(d) Is this set open in $\mathbb{R}$ (with the standard topology)?

(e) If we did choose a topology on $\mathbb{R}^\infty$ where U was an open set, explain why $\mathbb{R}^\infty$ with this topology would not satisfy the condition to be the product in the category of topological spaces.

Now let

$$V = (0,1) \times (0,1) \times \mathbb{R} \times (0,1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \cdots$$

i.e., $V = \prod_{n \in I} B_n$ where $B_n = (0,1)$ for $n \in \{1,2,4\}$, and $B_n = \mathbb{R}$ otherwise.

(f) Find $u^{-1}(V)$. Is $u^{-1}(V)$ an open set in $\mathbb{R}$?

3. Let $X = \{0, 1\}$ with the discrete topology, for each $i \in \mathbb{N}$ set $X_i = X$, and let $P = \prod_{i\in\mathbb{N}} X_i$ be the product set. We can think of an element $\underline{x} \in P$ as an infinite sequence of 0's and 1's, $\underline{x} = (x_0, x_1, x_2, x_3, \ldots)$.

Recall that the product topology (in this case, where all X_i are the same X) has a base of the form

$$\mathcal{B} = \left\{ \prod_{i\in\mathbb{N}} B_i \,\middle|\, B_i \in \tau_X, \text{ for all } i, \text{ and } B_i = X \text{ for all but finitely many } i \right\}.$$

We also have the box topology, with base

$$\mathcal{B}' = \left\{ \prod_{i\in\mathbb{N}} B_i \,\middle|\, B_i \in \tau_X, \text{ for all } i \right\}.$$

(a) Show that the box topology on P is the discrete topology. In particular, show for each $\underline{x} \in P$ that the set $\{\underline{x}\}$ is open.

Now let us put the product topology[1] on P. Let $U \subseteq P$ be a nonempty open set, and $\underline{x} \in U$.

(b) Show that there is a finite set $J \subseteq \mathbb{N}$ so that the set

$$V = \left\{ \underline{y} = (y_0, y_1, y_2, \ldots) \in P \,\middle|\, y_j = x_j \text{ for all } j \in J \right\}$$

is contained in U. Also show that V is an open set.

(c) Conclude, in particular, that in the product topology on P all nonempty open sets are infinite, and so on P the box topology and product topology are different topologies.

[1] . . . the right topology on a product of topological spaces . . .