DUE DATE: OCT. 25, 2024

1. Let X be a topological space and R an equivalence relation on X.

(a) What does it mean that a pair (W, π) (where W is a topological space, and $\pi\colon X \longrightarrow W$ a continuous map) is a quotient of X by R? (I.e., what is the definition of quotient topological space?)

Suppose that (W, π) and (W', π') are both quotients of X by R. Show that (W, π) and (W', π') are isomorphic by unique isomorphisms compatible with their role as quotients. Specifically, show that

(b) there is a unique morphism $f\colon W \longrightarrow W'$ so that $\pi' = f \circ \pi$, a unique morphism $g\colon W' \longrightarrow W$ so that $\pi = g \circ \pi'$, that $g \circ f = \mathrm{Id}_W$, and that $f \circ g = \mathrm{Id}_{W'}$.

2. Let $X = [0, 1]$ and let R be the equivalence relation on X such that $0 \sim 1$, and such that a point $x \in (0, 1)$ is only equivalent to itself. In this question we will confirm that X/R is the circle

$$S^1 = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \right\} \subset \mathbb{R}^2$$

with its standard topology (i.e., with the subspace topology inherited from the standard topology on $\mathbb{R}^2$).

Let $(X/R, \pi)$ be the quotient of X by R, and let $f\colon X \longrightarrow S^1$ be the map $f(x) = (\cos(2\pi x), \sin(2\pi x))$. (This π is the number π, not the quotient map.)

(a) Is the map f continuous? What is the simplest argument showing this?

(You may assume that cos and sin are continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Your answer should probably involve both the universal property of the product, and something about the subspace topology.)

(b) Explain how to use f to get a continuous map $g\colon X/R \longrightarrow S^1$ and why this map is a bijection.

Thanks to the fact that g is a bijection, we may consider X/R (as a set) to be S^1, and what is left is to show that the topology on X/R — now called S^1 — is the standard topology on S^1. Let τ_Q be the topology on S^1 coming from the construction as a quotient, and τ_S the standard topology on S^1.

(c) Explain why τ_Q is a finer topology than τ_S.

(d) Show that $\tau_Q = \tau_S$ (and thus that the quotient really is S^1 with the standard topology).

SUGGESTION : By (c) we have $\tau_S \subseteq \tau_Q$, so one possible strategy for (d) is to show the opposite inclusion, and one way to do that is to pick a convenient base for τ_Q and show that for every element U in that base, $U \in \tau_S$. Finally, to understand the open sets in τ_Q (and so determine a convenient base for your argument), why not use the fact that open sets in τ_Q correspond to R-saturated open sets in X?

3. Let $X = \mathbb{R}^2$, and consider the following equivalence relation R on X :

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } \exists\, t \in \mathbb{R} \text{ such that } (x_2, y_2) = (x_1 + ty_1, y_1).$$

We may also describe the equivalence relation using matrix notation. The equation $(x_2, y_2) = (x_1 + ty_1, y_1)$ can be written as

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

(a) What (geometrically) are the equivalence classes under this equivalence relation?

You may want to consider the case that a point is off the x-axis, or on the x-axis, separately.

Let $(X/R, \pi)$ be the quotient space. We will now try and determine X/R with its quotient topology.

Let Z be the union of of the x- and y-axes in $\mathbb{R}^2$, and τ_S the subspace topology on Z. Let $h\colon Z \longrightarrow X/R$ be the composition of the inclusion i_Z and π.

(b) Explain why h is a (continuous) bijection.

By (b) we may identify Z and X/R as sets. Let τ_Q be the quotient topology on Z (i.e., the one coming from our bijection of Z and X/R). By (b), $\tau_Q \subseteq \tau_S$.

Let $W \subseteq Z$ be the y-axis minus $(0, 0)$. The closure of W (in Z) in the standard topology is just the y-axis, but this is not the closure of W in the quotient topology.

(c) Find the closure of W in the quotient topology on Z.

(This might be a good time to use the bijection between closed sets in the quotient topology and R-saturated closed subsets of $\mathbb{R}^2$.)

The lesson we learn from (c) is that τ_Q is a proper subset of τ_S – not all sets which are open in τ_S are open in τ_Q. Let us now try and find a base for the topology τ_Q. We can do this by taking elements of a base for the topology in $\mathbb{R}^2$, considering the R-saturation of those sets, and then, if the saturations are open, restricting to Z.

Let $(0, y) \in Z$ be a point on the y-axis, $y \neq 0$, and let $B_\delta(0, y) \subset \mathbb{R}^2$ be an open ball around $(0, y)$ of radius δ, with $\delta < |y|$ (so that $B_\delta(0, y)$ does not contain $(0, 0)$).

(d) What is the R-saturation of $B_\delta(0, y)$? Is it open? What is the restriction of this set to Z?

(e) Conclude from (d) that any open interval on the y-axis not containing $(0, 0)$ is open in τ_Q.

Similarly, let $(x, 0) \in Z$ be a point on the x-axis (allowing $x = 0$), and let $B_\delta(x, 0) \subset \mathbb{R}^2$ be an open ball around $(x, 0)$ of radius δ (with no restriction on δ, other than $\delta > 0$).

(f) What is the R-saturation of $B_\delta(x, 0)$? Is it open? What is the restriction of this set to Z?

For any $\delta > 0$, let

$$V_\delta = \left\{(0, y) \,\middle|\, |y| < \delta,\ y \neq 0\right\} \subseteq Z.$$

(Note that by (e) V_δ is an open subset in τ_Q.)

(g) Conclude from (f) that if a subset $U \subseteq Z$ is open in τ_Q, and if U contains a point on the x-axis, then U must contain a subset of the form V_δ for some $\delta > 0$.

(h) Given an open subset $U \in \tau_S$, when is $U \in \tau_Q$?