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1. Suppose that X is a T_1 topological space (i.e., that every point is closed), and let $A = \{x_1, \ldots, x_n\} \subseteq X$ be a finite set of points.

(a) Show that A is closed in X.

(b) Show that (in the subspace topology) A is a discrete topological space.

Now we drop the assumption that X is a T_1 topological space, and suppose instead that $x_1, \ldots, x_{n+1}$ are distinct points of X, that we have open sets U_i', $i = 1, \ldots, n$ so that $x_i \in U_i'$ for each i, and that $U_i' \cap U_j' = \varnothing$ if $i \neq j$. We also suppose that for each $i = 1, \ldots, n$ we have open sets V_i and W_i with $x_i \in V_i$ and $x_{n+1} \in W_i$, and such that $V_i \cap W_i = \varnothing$.

For each i, $i = 1, \ldots, n$, let $U_i = U_i' \cap V_i$, and set $U_{n+1} = \bigcap_{i=1}^{n} W_i$.

(c) Show that $U_1, \ldots, U_{n+1}$ are open sets, that $x_i \in U_i$ for each i, $i = 1, \ldots n+1$, and that $U_i \cap U_j = \varnothing$ for each $i \neq j$.

(d) Suppose that X is a Hausdorff topological space, and that $x_1, \ldots, x_n \in X$ are distinct points. Show that there are open sets $U_1, \ldots, U_n \subseteq X$ with $x_i \in U_i$ for each i, and such that $U_i \cap U_j = \varnothing$ if $i \neq j$.

2. Suppose that X and Y are Hausdorff topological spaces. In this problem we will show that $X \times Y$ is also a Hausdorff topological space. This result is valid for an arbitrary number of factors (i.e., if each X_i is Hausdorff, then $\prod_{i \in I} X_i$ is Hausdorff, even when I is infinite), but to reduce notation we will only show the case that $|I| = 2$.

(a) Show directly that $X \times Y$ satisfies the condition to be Hausdorff. I.e., if (x_1, y_1) and (x_2, y_2) are distinct points of $X \times Y$, show that there are open sets U_1 and U_2 with $(x_1, y_1) \in U_1$, $(x_2, y_2) \in U_2$, and $U_1 \cap U_2 = \varnothing$.

Next, let us prove the same result using our other characterization of a Hausdorff topological space, that the diagonal is closed. Recall that the product is associative (up to unique isomorphism) so that $(X \times Y) \times (X \times Y) = X \times Y \times X \times Y$.

Let $q_1 \colon X \times Y \times X \times Y \longrightarrow X \times X$ and $q_2 \colon X \times Y \times X \times Y \longrightarrow Y \times Y$ be the maps $q_1(x_1, y_1, x_2, y_2) = (x_1, x_2)$ and $q_2(x_1, y_1, x_2, y_2) = (y_1, y_2)$ respectively.

(b) Show that $\Delta_{X \times Y} = q_1^{-1}(\Delta_X) \cap q_2^{-1}(\Delta_Y)$.

(c) Using (b), give another argument that if both X and Y are Hausdorff then $X \times Y$ is Hausdorff.

3. Let $X = \mathbb{R}^2$ with the equivalence relation R from **H6 Q3**. In this question we will check (using your answers from **H6**) that X/R is not Hausdorff. Recall in that question we have identified X/R with the union of the x- and y-axes, but with a topology different from the standard one. We will use the notation of points on the union of the x- and y-axes to talk about points in X/R.

(a) Show that if $(0, y_1)$, $(0, y_2)$ are distinct points of X/R with y_1, $y_2 \neq 0$, then there are open sets U_1, $U_2 \subseteq X/R$ with $(0, y_i) \in U_i$, $i = 1$, 2, and $U_1 \cap U_2 = \varnothing$.

(b) Similarly, if $(0, y)$ and $(x, 0)$ are points of X/R, with $y \neq 0$, show that $(0, y)$ and $(x, 0)$ can be separated by open sets as in (a).

(c) Finally, suppose that $(x_1, 0)$, $(x_2, 0)$ are distinct points of X/R. Show that if U_1, U_2 are open sets with $(x_i, 0) \in U_i$, $i = 1$, 2, then $U_1 \cap U_2 \neq \varnothing$. Thus, X/R is not a Hausdorff topological space.

4. Let X be a set, and R and equivalence relation on X. The *graph of* R is the subset

$$\Gamma_R = \left\{ (x_1, x_2) \in X \times X \,\middle|\, x_1 \sim_R x_2 \right\} \subseteq X \times X.$$

Note that even though the name "graph" is used, this is not the graph of a function unless R is a trivial relation. (For the graph $\Gamma_f \subseteq X \times Y$ of a function $f \colon X \longrightarrow Y$, projection onto X gives a bijection between Γ_f and X, and this does not happen in general for $\Gamma_R \subseteq X \times X$).

Let $\pi \colon X \longrightarrow X/R$ be the quotient map, and $\pi \times \pi \colon X \times X \longrightarrow (X/R) \times (X/R)$ the corresponding map of products (i.e., so that $(\pi \times \pi)(x_1, x_2) = (\pi(x_1), \pi(x_2))$).

(a) Show that $\Gamma_R = (\pi \times \pi)^{-1}(\Delta_{X/R})$.

Now assume that X is a topological space. It is useful to know when the quotient X/R is Hausdorff.

(b) Prove the one-way implication that if X/R is Hausdorff then Γ_R is closed in $X \times X$.

It would be very convenient if (b) were a two-way implication, but in general it is not. (I find the reason that it is not a two-way implication a somewhat subtle issue.) One additional condition which does allow the opposite implication (i.e, that Γ_R closed $\Longrightarrow$ X/R Hausdorff) is that π be an *open* map, that is, if for every open set $U \subset X$, $\pi(U)$ is open in X/R. This openness condition is automatic if the equivalence relation is given by the orbits of a group action, where each element of the group acts continuously, a common setting for the quotient construction.

Now let us go back to two of our quotient examples, which we know to be Hausdorff and non-Hausdorff respectively, and check the condition that Γ_R is closed.

To start with, let $X = \mathbb{R}^2 \setminus \{(0,0)\}$, with the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ if and only if $\exists\ t \in \mathbb{R}_{>0}$ such that $(x_2\,, y_2) = (tx_1,\, ty_1)$. In this case we know that $X/R = S^1$, which is certainly a Hausdorff topological space.

(c) Show that $(x_1, y_1) \sim (x_2, y_2)$ if and only if

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0,\ \ x_1 x_2 \geqslant 0, \text{ and } y_1 y_2 \geqslant 0.$$

(The first equation is that the determinant of the 2×2 matrix is 0.)

(d) Show that Γ_R is closed in $X \times X$.

Now let $X = \mathbb{R}^2$ with the relation R given in **H6 Q3**. We have checked in **Q3** of this assignment that the quotient X/R is not Hausdorff. Note that $X \times X = \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. One "piece" of Γ_R, associated to the equivalence classes $[(x, 0)]$ for $x \in \mathbb{R}$ is the subset

$$\Big\{(x, 0, x, 0) \,\Big|\, x \in \mathbb{R}\Big\} \subseteq \mathbb{R}^4.$$

(e) Find the other part of Γ_R (associated to the equivalence classes of points (x, y), $y \neq 0$).

(f) Show that Γ_R is not closed by finding a point in $\overline{\Gamma}_R$ which is not in Γ_R.

For (f), it is enough to give a specific point in $\mathbb{R}^4$ which you claim is in $\overline{\Gamma}_R$, some reason why you know/think it is in $\overline{\Gamma}_R$ (e.g., a sequence of points in Γ_R which converge to your point) and a reason why that point is not in Γ_R, and so in $\overline{\Gamma}_R \setminus \Gamma_R$.