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1. In this problem we will study the idea of *connected components* of a topological space. Let X be a topological space, $x_0 \in X$, and set

$$C_{x_0} = \bigcup_{\substack{A \subseteq X \text{ connected} \\ x_0 \in A}} A.$$

Here the union is over all connected subsets $A \subseteq X$ containing x_0.

The set C_{x_0} is called the *connected component of X containing x_0*. The notation C_{x_0} is not standard, and there does not seem to be a standard notation for the connected component. (Note that this is different from the symbol $\complement_{x_0}$.)

Give brief arguments (and they should only require brief arguments) to justify each of the following statements.

(a) C_{x_0} is connected.

(b) C_{x_0} is maximal with respect to the properties of containing x_0 and being connected. (That is, if A is any connected set containing x_0, then $A \subseteq C_{x_0}$.)

This characterization will be useful in the following parts.

(c) Show that C_{x_0} is closed.

Furthermore, show that

(d) For any connected subset $A \subseteq X$, either $A \cap C_{x_0} = \varnothing$ or $A \subseteq C_{x_0}$.

(e) If $x_1 \in X$ is any other point, then either $C_{x_0} \cap C_{x_1} = \varnothing$ or $C_{x_0} = C_{x_1}$.

Note that by (e) the connected components partition X, and that $C_{x_0} = C_{x_1}$ if and only if $x_1 \in C_{x_0}$ (and symmetrically that $x_0 \in C_{x_1}$).

You may wonder if there is such a thing as "the set of connected components of X", or even . . . a *space* of connected components of X. Well, good news!

Part (e) also shows that the relation $x_0 \sim_R x_1$ if and only if $C_{x_0} = C_{x_1}$ (i.e., if and only if x_0 and x_1 are in the same connected component of X) is an equivalence relation, with the equivalence classes being the connected components. We can therefore form the quotient space X/R.

(f) Explain briefly why we know that X/R is a T_1 space.

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If X/R is finite, this means that X/R has the discrete topology. It can even happen that X/R has the discrete topology when X/R is infinite. But, considering the case of $\mathbb{Q}$ shows that this is not always the case. In $\mathbb{Q}$, every point is its own connected component, and thus on $\mathbb{Q}$ the equivalence relation above is trivial, and so $\mathbb{Q}/R = \mathbb{Q}$, which does not have the discrete topology.

The answer in general (to the question "what can we say about the topology of the space of connected components?") is that, with the relation above, X/R is always a totally disconnected topological space. In the rest of the problem we will work out a proof of this fact.

To show that X/R is totally disconnected, we need to show that every subset $B \subset X/R$ with two or more elements is disconnected.

(g) Explain why it suffices to show that every *closed* subset $B \subset X/R$ with two or more elements is disconnected. (That is, show that if we know that every closed subset with two or more elements is disconnected, then this implies that every subset with two or more elements is disconnected.)

Why does this help? As we will see below, it will simplify part of the argument to know that B is closed. Let $\pi \colon X \longrightarrow X/R$ be the quotient map, and $D = \pi^{-1}(B)$. Since B is closed, D is closed, and since B has at least two elements, D contains at least two connected components of X.

(h) Explain why we know that D must be disconnected.

Thus, D can be written as a disjoint union $D = D_1 \sqcup D_2$, with each D_i closed in D.

(i) Explain why we know that D_1 and D_2 are closed in X.

(j) For each connected component C (of X) contained in D, explain why either $C \subseteq D_1$ or $C \subseteq D_2$.

Thus, D_1 and D_2 are each R-saturated closed subsets of X and so correspond to two closed subsets B_1 and B_2 of X/R.

(k) Complete the proof by explaining why B is disconnected.

2. In this problem we will give a shorter argument for the conclusion of **H6 Q2**.

(a) Suppose that X is a quasi-compact topological space, and that R is an equivalence relation on X. Show that X/R is quasi-compact.

Now let us return to the situation of **H6 Q2**. Let $X = [0, 1]$, and let R be the equivalence relation such that $0 \sim_R 1$, and such that points in $(0, 1)$ are only equivalent to themselves. Recall that in that problem we have constructed a continuous bijection $g \colon X/R \longrightarrow S^1$.

(b) Explain how we know that S^1 is Hausdorff.

(c) Show that X/R is homeomorphic to S^1.

3. In this problem we will construct some examples of quasi-compact subspaces which are not closed in the larger topological space. We will work with the "arrow", the topology on $X = \mathbb{R}$ where the open sets are the sets (a, ∞), and so closed sets are the sets $(-\infty, a]$ (and $\varnothing$ and X of course). This topological space appeared in **H1 Q1**.

Let $A \subseteq X$ be a subset, and let $x_0 = \inf A$ (i.e., the usual infimum of a set $A \subseteq \mathbb{R}$).

(a) Show that if $x_0 \in A$ then A is quasi-compact.

(b) Conversely, if $x_0 \notin A$, show how to construct an open cover of A which does not have a finite subcover.

Thus, combining (a) and (b), a subset $A \subseteq X$ is quasi-compact if and only if it contains its own infimum.

(c) Use this to construct a subspace $A \subseteq X$ which is quasi-compact but not closed.

(d) Also construct an example as in (c), but with a subspace $B \subseteq A$ which is again quasi-compact, and not closed in A.

Thus, in (d) we have constructed an example of a topological space which is quasi-compact, and which has a non-closed quasi-compact subspace.

(In class we showed that a closed subset of a quasi-compact space is always quasi-compact. The purpose of this example is to show that in general this is a one-way implication. In contrast, when the topological space is Hausdorff, we know that the implication is two-way.)