DUE DATE: Nov. 15, 2024

1. In class we have shown that a subset $A \subseteq \mathbb{R}$ (in the usual topology) is compact if and only if A is closed and bounded. The same characterization is true for compact subsets of $\mathbb{R}^n$, and in this question we will use results from class to prove this. (This result is called the Heine-Borel theorem, and you may have also seen it in Math 281.) Each of these steps should be short, and most rely on just applying a result from class.

Since $\mathbb{R}^n$ is Hausdorff, any subset $A \subseteq \mathbb{R}^n$ is also Hausdorff, so for subsets of $\mathbb{R}^n$ being quasi-compact is the same as being compact.

First suppose that $A \subseteq \mathbb{R}^n$ is compact.

(a) Explain why we know that A must be closed (in $\mathbb{R}^n$).

For each i, $i = 1, \ldots, n$, let $p_i \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be projection onto the i-th coordinate, and let $f_i = p_i|_A$ (i.e., f_i is p_i restricted to A).

(b) Explain why f_i must achieve a maximum and minimum value on A.

(c) Explain why we know that A is bounded. (It might help to write out what it means for a subset of $\mathbb{R}^n$ to be bounded.)

Conversely, let us assume that $A \subseteq \mathbb{R}^n$ is closed and bounded.

(d) Explain why this means that A must be contained in a product of closed intervals

$$[a_1, b_1] \times [a_2, b_n] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n.$$

(e) Explain why, to show that A is compact, it suffices to show that the product set $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact.

(f) Explain why we know that this product of intervals is compact.

2. Let $X = [-1, 3]$ and define a family of open subsets U_t of X, for $t \in [0, 1]$ by

$$U_t = \begin{cases} [-1, 2t) & \text{if } t > \frac{1}{2} \\ [-1, 1/2) & \text{if } t = \frac{1}{2}. \\ [-1, t/2) & \text{if } t < \frac{1}{2}. \end{cases}$$

These subsets satisfy the condition needed for the proof of Urysohn's lemma : If $t_1 < t_2$ then $\overline{U}_{t_1} \subseteq U_{t_2}$ (you do not need to verify this). As in the proof of Urysohn's lemma, define a function $f\colon X \longrightarrow [0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \notin U(1) \\ \inf\{t \mid x \in U_t\} & \text{otherwise.} \end{cases}$$

(a) Draw the graph of f. (The function f is a continuous function from $[-1, 3]$ to $[0, 1]$. It has a graph, à la MATH 120!)

As part of the proof of Urysohn's lemma, we defined[1] the subsets

$$V_{t_2} = \bigcup_{\epsilon>0} U_{t_2-\epsilon} \quad \text{and} \quad Z_{t_1} = \bigcap_{\epsilon>0} \overline{U}_{t_1+\epsilon}.$$

It may not have been clear why we did not use U_{t_2} or $\overline{U}_{t_1}$ instead. Let us think about what these sets are in this example.

(b) Find $V_{\frac{1}{2}}$.

(c) Is $f(x) < \frac{1}{2}$ for all $x \in V_{\frac{1}{2}}$?

(d) What is $U_{\frac{1}{2}} \setminus V_{\frac{1}{2}}$?

(e) What value does f have on the points in (d)?

(f) What is the set $\{x \in X \mid f(x) < \frac{1}{2}\}$?

(g) Find $Z_{\frac{1}{2}}$.

(h) Is $f(x) \leqslant \frac{1}{2}$ for all $z \in Z_{\frac{1}{2}}$?

(i) What is $Z_{\frac{1}{2}} \setminus \overline{U}_{\frac{1}{2}}$?

(j) What value does f have on the points in (i)?

(k) What is the set $\{x \in X \mid f(x) > \frac{1}{2}\}$?

[1] If $t_2 - \epsilon < 0$ then $U_{t_2-\epsilon}$ should be interpreted as $\varnothing$, and if $t_1 + \epsilon > 1$ then $U_{t_1+\epsilon}$ should be interpreted as X. In any case, both definitions really only depend on ϵ as $\epsilon \to 0^+$, and for small enough ϵ there is no confusion.