Math 347 — Homework Assignment 10

DUE DATE: MAR. 27, 2026

1. Let us consider the topology on $\mathbb{Z}$ used in the proof that there are infinitely many primes, namely, the one with base

$$\left\{ B_n(a) \,\middle|\, a,\, n \in \mathbb{Z},\, n \neq 0 \right\},$$

where

$$B_n(a) = \left\{ a + kn \,\middle|\, k \in \mathbb{Z} \right\} = \left\{ b \in \mathbb{Z} \,\middle|\, b \equiv a \bmod n \right\}.$$

It is not clear if we should think of this as an actual topology, but let us use this as an excuse to think about some of the properties we know.

(a) Show that $\mathbb{Z}$ (with this topology) is Hausdorff.

(b) Show that $\mathbb{Z}$ is totally disconnected. (From the class of Feb. 26th.)

(c) Prove that the addition map $A\colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is continuous. (The addition map sends (x, y) to $x + y$).

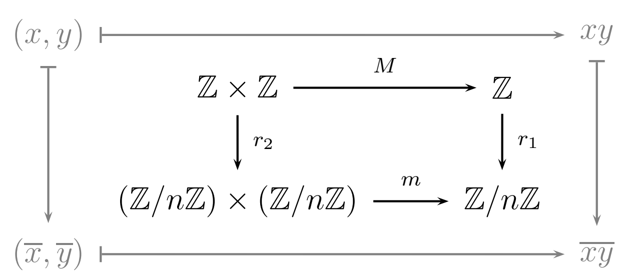
Here (of course!) $\mathbb{Z} \times \mathbb{Z}$ has the product topology, using our topology on $\mathbb{Z}$. Since we can check continuity by using elements of a base, concretely this means, for every set $B_n(a)$ in our base, we need to show that the inverse image of $B_n(a)$ under addition, i.e., the set

$$A^{-1}\left(B_n(a)\right) = \left\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, x + y \in B_n(a) \right\}$$

is open in $\mathbb{Z} \times \mathbb{Z}$. To show that, we need to show that this set can be covered by open sets of the form $B_{n_1}(a_1) \times B_{n_2}(a_2)$.

(d) Similarly, prove that the multiplication map $M\colon \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is continuous. (The multiplication map sends (x, y) to xy)

NOTE : For (d), for a fixed n it may help to consider the diagram at right, where the vertical maps are reduction mod n, and the horizontal maps are multiplication. The diagram commutes since reduction mod n is a ring homomorphism.



In this diagram the part in black is the diagram of sets (and topological spaces), while the parts in grey around the outside show what happens to the elements.

The inverse inverse image of $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ under the map r_1 is exactly the open set $B_n(a)$, and so if we give $\mathbb{Z}/n\mathbb{Z}$ the discrete topology, then the "reduction mod n" maps are continuous (because the inverse image of $\overline{a}$ is $B_n(a)$), and the multiplication map $m\colon (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathbb{Z}/n\mathbb{Z}$ is continuous, because every map of sets where the source has the discrete topology is continuous.

It should be possible to put all that together to show that the multiplication map is continuous.

Or, you can also argue directly from the definition of the topology, as in part (c).

(NOTE : The 'diagram idea', replacing multiplication with addition, can also be used to answer (c).)

2. Let $\max\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function sending each $(x, y) \in \mathbb{R}^2$ to $\max(x, y)$.

(a) Show that max is a continuous function on $\mathbb{R}^2$ by covering $\mathbb{R}^2$ by two closed subsets Z_1 and Z_2, in such a way that $\max|_{Z_1}$ and $\max|_{Z_2}$ are obviously continuous.

(b) Let X be a topological space, and $f, g\colon X \longrightarrow \mathbb{R}$ continuous functions. Show that the function $x \mapsto \max(f(x), g(x))$ is a continuous function on X.

3. Suppose that γ_a, γ_b, and γ_c are three paths in a topological space X, with $\gamma_a(1) = \gamma_b(0)$, and $\gamma_b(1) = \gamma_c(0)$. Let $(\gamma_a \circledast \gamma_b \circledast \gamma_c)$ be the path defined by

$$(\gamma_a \circledast \gamma_b \circledast \gamma_c)(s) = \begin{cases} \gamma_a(3s) & \text{if } s \in [0, \frac{1}{3}] \\ \gamma_b(3s - 1) & \text{if } s \in [\frac{1}{3}, \frac{2}{3}] \\ \gamma_c(3s - 2) & \text{if } s \in [\frac{2}{3}, 1] \end{cases} \qquad \text{for } s \in [0, 1].$$

Find a path homotopy between $(\gamma_a \circledast \gamma_b \circledast \gamma_c)$ and $(\gamma_a * \gamma_b) * \gamma_c$.

NOTES:

(1) $(\gamma_a \circledast \gamma_b \circledast \gamma_c)$ is a completely made-up notation for this question.

(2) One purpose of this question could be that, if we wrote down a similar homotopy between $\gamma_a * (\gamma_b * \gamma_c)$ and $(\gamma_a \circledast \gamma_b \circledast \gamma_c)$, then by combining the homotopies (i.e., by transitivity of the equivalence relation of path homotopy) we would have another argument that $\gamma_a * (\gamma_b * \gamma_c) \sim (\gamma_a * \gamma_b) * \gamma_c$.

(3) But, the real purpose of this question is for you to practice writing down a homotopy of this kind.