DUE DATE: APR. 4, 2026

1.

(a) Suppose that X is a topological space, and $A \subseteq X$ a subspace. What is the definition of a "retract of X onto A"?

(b) Let S^1 be the equator of S^2, with inclusion map $i(x,y) = (x,y,0)$. In the class of January 27th ("Functors") it was claimed that there is no retract $f\colon S^2 \longrightarrow S^1$. Prove this assertion.

SUGGESTION : Use an argument similar to our argument that there is no continuous retract $D^2 \longrightarrow S^1$.

(c) Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories. What is functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$? (That is, what data does one have to give in order to give a functor? What properties must F have with respect to this data?)

(d) In your argument in (b), which properties of a functor are used? For instance, if the functor π_1 did not satisfy one of the properties listed in (c), would the proof in (b) still work?

2. Let $U = S^2 \setminus \{(0,0,-1)\}$. In class we have argued that since U is homeomorphic to $\mathbb{R}^2$, and since we can contract every loop in $\mathbb{R}^2$ to a point, then we can contract every loop in U to a point. This argument is correct!

But, we could also try and give a direct construction contracting every loop in U to a point (say to $(0,0,1)$).

(a) If $x^2+y^2+z^2=1$, show that the length of the vector $(1-t)(x,y,z)+t(0,0,1)$ is $\sqrt{(1-t)^2+2(1-t)tz+t^2}$.

Define a function $F\colon U \times [0,1] \longrightarrow \mathbb{R}^3$ by

$$F\big((x,y,z),t\big) = \frac{\big((1-t)x,\ (1-t)y,\ (1-t)z+t\big)}{\sqrt{(1-t)^2+2(1-t)tz+t^2}}.$$

(b) Show that the image of F lies in $U \subseteq \mathbb{R}^3$.

Thus, by restricting the codomain, we can consider F to be a function

$$F\colon U \times [0,1] \longrightarrow U.$$

For each $t \in [0,1]$, let $f_t\colon U \longrightarrow U$ be the function $F(\cdot, t)$, i.e., the function defined by $f_t(x,y,z) = F((x,y,z),t)$.

(c) Show that $f_0 = \mathrm{Id}_U$.

(d) Show that f_1 is the constant function $U \longrightarrow \{(0,0,1)\} \in U$.

(e) Suppose that $\gamma\colon [0,1] \longrightarrow U$ is a loop based at $x_0 = (0,0,1)$. Show that the map $H\colon [0,1] \times [0,1] \longrightarrow U$ defined by

$$H(s,t) = F(\gamma(s), t)$$

is a loop homotopy between γ and the constant loop. (Don't forget the "loop" part, i.e., the condition on the left and right edges of the square. Also don't forget to justify why H is continuous. You may assume that F is continuous.)

If it helps to understand what is happening in the formula in (e), H can also be described as the map $(s,t) \mapsto f_t(\gamma(s))$.

(f) Where in this argument did we use that $(0,0,-1) \notin U$? I.e., would this argument also apply to all of S^2 ?