DUE DATE: APR. 10, 2026

1. In this problem we will use the Borsuk-Ulam theorem to prove the following result (the "Lusternik-Schnirelmann theorem for S^2") :

> Let A_1, A_2, and $A_3 \subseteq S^2$ be closed subsets of S^2 with $A_1 \cup A_2 \cup A_3 = S^2$. Then there is at least one $i \in \{1, 2, 3\}$ such that A_i contains a pair of antipodal points.

As in class, for a point $x \in S^2$ let us use $\tilde{x}$ for the point antipodal to x. (You may also use $-x$ if you prefer.) If A_1 or A_2 contains a pair of antipodal points, we are done. So, let us assume that A_1 and A_2 contain no pair of antipodal points, and show that A_3 contains such a pair.

Let
$$B_1 = \left\{x \,\middle|\, \tilde{x} \in A_1\right\} \text{ and } B_2 = \left\{x \,\middle|\, \tilde{x} \in A_2\right\}.$$

Under our assumption $A_1 \cap B_1 = \varnothing$ and $A_2 \cap B_2 = \varnothing$. Define a function $f\colon S^2 \longrightarrow \mathbb{R}^2$ by
$$f(x) = \left(\frac{d(x, A_1)}{d(x, A_1) + d(x, B_1)}, \frac{d(x, A_2)}{d(x, A_2) + d(x, B_2)}\right).$$
where $d(\cdot, \cdot)$ denotes the distance in $\mathbb{R}^3$ (considering $S^2 \subseteq \mathbb{R}^3$ to be the unit sphere). Recall (from the notes on OnQ) that for a subset $C \subseteq \mathbb{R}^3$, $x \mapsto d(x, C)$ is a continuous function. Thus f is a continuous function from S^2 to $\mathbb{R}^2$.

(a) In order to define f, it is important that $A_1 \cap B_1 = \varnothing$ and that $A_2 \cap B_2 = \varnothing$. Why? (I.e., how does this assumption let us, or help us, define f?)

(b) For all $x \in S^2$, show that
$$f(x) + f(\tilde{x}) = (1, 1).$$

Applying the Borsuk-Ulam theorem, there is a point $x \in S^2$ such that $f(x) = f(\tilde{x})$.

(c) Show that x (for the point x above) is not in A_1 or A_2.

(d) Similarly show that $\tilde{x}$ is not in A_1 or A_2.

(e) Finish the proof of the theorem : show that x and $\tilde{x}$ are in A_3.

2. Suppose that $\varphi\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ is a group homomorphism, and set $m = \varphi(1)$. Since φ is a group homomorphism, it follows that $\varphi(k) = km$ for all $k \in \mathbb{Z}$. I.e., every group homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ is of the form "multiplication by m" for some m.

Now let $f\colon S^1 \longrightarrow S^1$ be a continuous function. Fixing $s_0 \in S^1$ (say our usual $s_0 = (1,0)$) we then obtain a group homomorphism

$$\begin{array}{cccc} f_* & : & \pi_1(S^1, s_0) & \longrightarrow & \pi_1(S^1, f(s_0)) \\ & & \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

By the above discussion this map must be multiplication by m for some $m \in \mathbb{Z}$. We call m the *degree of the map* f, and set $\deg(f) = m$.

In this problem we will prove the following : If $\deg(f) \neq 1$, then f has a fixed point.

Note that not all maps $f\colon S^1 \longrightarrow S^1$ have a fixed point. For instance, if f is rotation by θ, where θ is not a multiple of 2π (e.g., $\theta = \frac{\pi}{2}$, or $\theta = \pi$) then f has no fixed point. For such rotation maps $\deg(f) = 1$, and so this does not contradict the statement above.

In this problem it will be easier to think of S^1 as the unit circle in $\mathbb{C}$:

$$S^1 = \left\{ z \in \mathbb{C} \,\middle|\, |z| = 1 \right\}.$$

We will also therefore consider the universal covering map $p\colon \mathbb{R} \longrightarrow S^1$ to be

$$\begin{array}{ccccc} p & : & \mathbb{R} & \longrightarrow & S^1 \\ & & x & \longmapsto & e^{2\pi i x} \end{array}$$

We only need one more ingredient before we begin our argument : a way to define the winding number for loops which do not begin and end at $s_0 = 1 + 0i \in S^1$. Here is what we will use :

Given a loop $\gamma\colon [0,1] \longrightarrow S^1$ with basepoint $s_1 \in S^1$, let $x_1 \in \mathbb{R}$ be any point lying over s_1 (i.e., $x_1 \in p^{-1}(s_1)$), let $\tilde{\gamma}\colon [0,1] \longrightarrow \mathbb{R}$ be the lift of γ to $\mathbb{R}$ starting at x_1 (so $\tilde{\gamma}(0) = x_1$). Then $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an integer and we set $w(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$. (In the past we have lifted so that $\tilde{\gamma}(0) = 0$, so in those cases this produces the same number.)

Now let $f\colon S^1 \longrightarrow S^1$ be a continuous map, and let $\gamma\colon [0,1] \longrightarrow S^1$ be the map $s \mapsto e^{2\pi i s}$, i.e., the loop with winding number 1, so that $[\gamma]$ corresponds to 1 in $\pi_1(S^1, s_0) = \mathbb{Z}$.

To compute $m = \deg(f)$, we look at $f_*[\gamma] = [f \circ \gamma]$. Then $m = w(f \circ \gamma)$.

Set $s_1 = f(s_0)$, choose $x_1 \in p^{-1}(s_1)$, and let $\tilde{\gamma}$ be a lift of $f \circ \gamma$ to $\mathbb{R}$ starting at x_1. (If you like, you can use the more correct $\widetilde{f \circ \gamma}$ for this lift; it just seems a bit messier.) Thus $\tilde{\gamma}(0) = x_1$ and $\tilde{\gamma}(1) = x_1 + m$, with $m = \deg(f)$.

Next consider the map $g\colon S^1 \longrightarrow S^1$, given by $g(z) = f(z)/z$. Here we are using the group structure of S^1 : If $z \in S^1$, and $f(z) \in S^1$, then $f(z)/z \in S^1$.

One reason to consider g is that

(*i*) Showing that f has a fixed point is the same as showing that there is a $z \in S^1$ with $g(z) = 1$.

(a) Show that $g(s_0) = s_1$ (i.e., the same s_1 as before).

(b) Show that

$$s \mapsto \tilde{\gamma}(s) - s \qquad \text{for } s \in [0,1]$$

is a lift of the loop $g \circ \gamma$.

(c) Conclude that $\deg(g) = m - 1$

Thus

(*ii*) The map $g_*\colon \pi_1(S^1, s_0) \longrightarrow \pi_1(S^1, s_1)$ is multiplication by $m-1$.

Assume that f has no fixed point, so that $1 \notin \operatorname{Im}(g)$, and set $U = S^1 \setminus \{1\}$. The map g then factors as

$$S^1 \longrightarrow U \hookrightarrow S^1$$

and so g_* factors as

$$\pi_1(S^1, s_0) \longrightarrow \pi_1(U, s_1) \longrightarrow \pi_1(S_1, s_1).$$

(d) Show that $\pi_1(U, s_1) = 0$. (For instance, isn't U homeomorphic to an interval?)

(e) By combining (*d*) with (*ii*) above, conclude that

$$f \text{ has no fixed point} \implies m - 1 = 0.$$

The contrapositive of this statement is the one we wanted to prove. I.e., since $m = \deg(f)$, we have shown that $\deg(f) \neq 1 \implies f$ has a fixed point.