1. Suppose that $X \subset \mathbb{R}^n$ is a shape.

(a) If f_1 and f_2 are functions on $\mathbb{R}^n$, show that $f_1 = f_2$ on X (i.e., when restricted to X) if and only if $f_1 - f_2$ is zero on X.

(b) If g is a function on $\mathbb{R}^n$ which is zero when restricted to X, and h any function on $\mathbb{R}^n$, show that hg is zero when restricted to X.

(c) Now let X be the circle $\left\{(x,y) \,\middle|\, x^2+y^2=1\right\} \subset \mathbb{R}^2$. Take the following functions on $\mathbb{R}^2$ and organize them into groups according to their equality when restricted to X:

(**1**) 1; (**2**) y; (**3**) x^2+y^2; (**4**) x^2-y^2;

(**5**) $2x^2+1$; (**6**) $2x^2-1$; (**7**) x^4-y^4; (**8**) y^3+x^2y.

(I.e, group together the functions which are equal when restricted to X.)

[Math 813 only] (d) Let X be the unit circle as in part (c). Let $f(x,y)$ be any polynomial in x and y. Prove that there is a polynomial of the form $g(x,y) = g_0(x) + g_1(x)y$ such that the restriction of f to X is equal to the restriction of g to X.

Solution.

(a) Let $\psi\colon R[\mathbb{R}^n] \longrightarrow R[X]$ be the restriction map. We have shown in class that this map is a ring homomorphism. Saying that f_1 and f_2 are equal when restricted to X is the same as saying that $\psi(f_1) = \psi(f_2)$. Saying that $f_1 - f_2$ is the zero function when restricted to X is the same thing as saying that $\psi(f_1 - f_2) = 0$. However, since ψ is a ring homomorphism, we have $\psi(f_1 - f_2) = \psi(f_1) - \psi(f_2)$. Thus $0 = \psi(f_1 - f_2)$ is the same as $0 = \psi(f_1) - \psi(f_2)$, which is the same as $\psi(f_1) = \psi(f_2)$.

Alternate Solution. Set $g = f_1 - f_2$. If $f_1 = f_2$ when restricted to X, then $f_1(x) = f_2(x)$ for all $x \in X$ and hence $g(x) = f_1(x) - f_2(x) = 0$. Thus, if $f_1 = f_2$ when restricted to X, then g is the zero function on X. Conversely, if $g(x) = 0$ for all $x \in X$, then since $f_1 = f_2 + g$ we have $f_1(x) = f_2(x) + g(x) = f_2(x) + 0 = f_2(x)$ for all $x \in X$, and hence that $f_1 = f_2$ when restricted to X.

(b) Let ψ be the restriction homomorphism as in part (a). Then if $\psi(g) = 0$ we have $\psi(hg) = \psi(h)\psi(g) = \psi(h) \cdot 0 = 0$, and thus hg is zero when restricted to X.

Alternate Solution. For every $x \in X$ we have $(fg)(x) = f(x)g(x) = f(x) \cdot 0 = 0$, and thus fg is also the zero function when restricted to X.

(c) The groups are:

A. $\boxed{1,\ x^2+y^2}$ **B.** $\boxed{2x^2+1}$

C. $\boxed{x^2-y^2,\ 2x^2-1,\ x^4-y^4}$ **D.** $\boxed{y,\ y^3+x^2y}$

We first check that the functions in each group have the same restriction to X, and then check that functions in different groups have different restrictions to X.

A: It is clear that $1 = x^2+y^2$ on X, this is the defining equation of X!

B: There is nothing to check: there is only one function in group **B**.

C: Starting with $x^2+y^2 = 1$ and multiplying both sides by x^2-y^2 we get

$$x^4-y^4 = (x^2-y^2)(x^2+y^2) = (x^2-y^2)\cdot 1 = x^2-y^2.$$

Starting with $x^2+y^2-1 = 0$ and adding x^2-y^2 to both sides gives

$$x^2-y^2 = x^2-y^2+0 = x^2-y^2+\left(x^2+y^2-1\right) = 2x^2-1.$$

Thus x^2-y^2, $2x^2-1$, and x^4-y^4 are all the same function when restricted to X.

D: Starting with $x^2+y^2=1$ and multiplying both sides by y gives

$$y = y\cdot 1 = y(x^2+y^2) = y^3+x^2y.$$

Therefore y and y^3+x^2y are the same function when restricted to X.

We now need to check that none of these groups should be joined, i.e., that these four different groups really give four different functions when restricted to X. Evaluating the functions at the point $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) \in X$ we get the values 1, 2, 0, and $\frac{1}{\sqrt{2}}$ for **A**, **B**, **C**, and **D** respectively. Since these numbers are all different, the functions in the different boxes are all different when restricted to X.

[Math 813 only] (d) On X we have the relation $x^2+y^2=1$ or $y^2 = 1-x^2$. Given any polynomial $f(x,y)$ in x and y, write $f(x,y) = \sum_{i,j\geq 0} c_{i,j}x^iy^j$ with the $c_{i,j}\in k$. We can then regroup the terms depending on whether j is even or odd:

$$\begin{aligned} f(x,y) &= \sum_{\substack{i,j\geq 0\\ j \text{ even}}} c_{i,j}x^iy^j + \sum_{\substack{i,j\geq 0\\ j \text{ odd}}} c_{i,j}x^iy^j \\ &= \sum_{\substack{i,j\geq 0\\ j \text{ even}}} c_{i,j}x^iy^j + y\sum_{\substack{i,j\geq 0\\ j \text{ odd}}} c_{i,j}x^iy^{j-1} \\ &= \sum_{\substack{i,j\geq 0\\ j \text{ even}}} c_{i,j}x^i(y^2)^{\frac{j}{2}} + y\sum_{\substack{i,j\geq 0\\ j \text{ odd}}} c_{i,j}x^i(y^2)^{\frac{j-1}{2}} \end{aligned}$$

Now set

$$g(x,y) = \sum_{\substack{i,j\geq 0\\ j \text{ even}}} c_{i,j}x^i(1-x^2)^{\frac{j}{2}} + y\sum_{\substack{i,j\geq 0\\ j \text{ odd}}} c_{i,j}x^i(1-x^2)^{\frac{j-1}{2}}.$$

Then $g(x,y)$ is of the required form, and since $y^2 = (1-x^2)$ on X, the restriction of g to X is the same as the restriction of f to X.

2. Let X be the unit circle $\{(x,y) \mid x^2+y^2 = 1\} \subset \mathbb{R}^2$ and Y the unit sphere $\{(u,v,w) \mid u^2+v^2+w^2 = 1\} \subset \mathbb{R}^3$. Define a map $\varphi\colon X \longrightarrow Y$ by the rule $\varphi(x,y) = (xy, y^2, x)$.

(a) Show that φ is well-defined. That is, show that if $(x,y) \in X$ then $\varphi(x,y) \in Y$.

(b) Compute $\varphi^*(u)$, $\varphi^*(v)$, and $\varphi^*(w)$.

(c) Compute $\varphi^*(3u^2 - 2vw + 5)$.

(d) Let f be the function $5xy^3+7x^2-9y^2$ restricted to X. Find a polynomial $g(u,v,w)$ on $\mathbb{R}^3$ so that $f = \varphi^*(g)$.

Solution.

(a) If $(x,y) \in X$ then $x^2+y^2 = 1$ and so

$$(xy)^2 + (y^2)^2 + (x)^2 = x^2y^2 + y^4 + x^2 = y^2(x^2+y^2) + x^2 = y^2\cdot 1 + x^2 = x^2+y^2 = 1.$$

Therefore if $(x,y) \in X$, $(xy, y^2, x) \in Y$.

(b) The functions u, v, and w are the coordinate functions on $\mathbb{R}^3$. For any $(x,y) \in X$ we therefore have

$$\begin{aligned}
(\varphi^*(u))(x,y) &= u(\varphi(x,y)) = u((xy,y^2,x)) = xy;\\
(\varphi^*(v))(x,y) &= v(\varphi(x,y)) = v((xy,y^2,x)) = y^2;\\
(\varphi^*(w))(x,y) &= w(\varphi(x,y)) = w((xy,y^2,x)) = x.
\end{aligned}$$

We thus conclude that $\varphi^*(u) = xy$, $\varphi^*(v) = y^2$, and $\varphi^*(w) = x$.

(c) Since φ^* is a ring homomorphism we have

$$\begin{aligned}
\varphi^*(3u^2 - 2vw + 5) &= 3\varphi^*(u)^2 - 2\varphi^*(v)\varphi^*(w) + \varphi^*(5)\\
&= 3(xy)^2 - 2(y^2)(x) + 5 = 3x^2y^2 - 2xy^2 + 5.
\end{aligned}$$

(d) From part (b) and the fact that φ^* is a ring homomorphism we see that

$$\begin{aligned}\varphi^*(5uv + 7w^2 - 9v) &= 5\varphi^*(u)\varphi^*(v) + 7\varphi^*(w)^2 - 9\varphi^*(v) = 5(xy)(y^2) + 7(x)^2 - 9y^2 \\ &= 5xy^3 + 7x^2 - 9y^2\end{aligned}$$

on X. Therefore $g(u, v, w) = 5uv + 7w^2 - 9v$ works.

3. Let $X = \mathbb{R}$ and $Y = \mathbb{R}^2$. The ring of polynomial functions on X is $\mathbb{R}[x]$. The ring of polynomial functions on Y is $\mathbb{R}[x, y]$.

(a) The ring $\mathbb{R}[x]$ is a subring of $\mathbb{R}[x, y]$, i.e., the inclusion map $\psi_1 : \mathbb{R}[x] \longrightarrow \mathbb{R}[x, y]$ is a ring homomorphism. Find a map $\varphi_1 : Y \longrightarrow X$ such that pullback by φ_1 induces ψ_1. (I.e., "$\varphi_1^* = \psi_1$".)

(b) The map $\psi_2 : \mathbb{R}[x, y] \longrightarrow \mathbb{R}[x]$ given by "setting $y = 0$" (i.e., $\psi_2(f(x, y) = f(x, 0))$ is also a ring homomorphism. Find a map $\varphi_2 : X \longrightarrow Y$ so that $\varphi_2^* = \psi_2$.

(c) How would you describe these maps geometrically? (I.e., in a picture or in words, what do they do?)

Minor suggestion: The fact that there is more than one x may make things more confusing. Relabelling one set of variables and describing the ring homomorphisms in the new variables may make things a bit clearer.

Solution. Let us take the suggestion and write $\mathbb{R}[t]$ for the ring of polynomial functions on $\mathbb{R}$. Then the ring homomorphisms are given by

$$\boxed{\begin{array}{rcl}\psi_1 : \mathbb{R}[t] & \longrightarrow & \mathbb{R}[x, y] \\ g(t) & \longmapsto & g(x)\end{array}} \quad \text{and} \quad \boxed{\begin{array}{rcl}\psi_2 : \mathbb{R}[x, y] & \longrightarrow & \mathbb{R}[t] \\ f(x, y) & \longmapsto & f(t, 0)\end{array}}.$$

Substitute $t = x$ | Substitute $x = t$ and $y = 0$

In class we have seen that if X is any shape, and $\varphi : X \longrightarrow \mathbb{R}^n$ a map given by an n-tuple of functions $\varphi = (f_1, f_2, \ldots, f_n)$ then the pullback of the coordinate functions $z_1, \ldots, z_n$ on $\mathbb{R}^n$ is $\varphi^*(z_1) = f_1$, $\varphi^*(z_2) = f_2$, ..., $\varphi^*(z_n) = f_n$. In other words: the pullback of the coordinate functions tells us the map!

(a) We are looking for a map $\varphi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ so that $\varphi_1^* = \psi_1$. Since φ_1 maps to $\mathbb{R}$ it is given by a single function f. From the remarks above we see that f is the pullback of the coordinate function t on $\mathbb{R}$. If $\varphi_1^* = \psi_1$ then $f = \varphi_1^*(t) = \psi(t) = x$.

Therefore the only possible choice for φ_1 is $\varphi_1(x, y) = x$. The map φ_1^* is the homomorphism we are looking for. For any $g(t) \in \mathbb{R}[t]$,

$$\varphi_1^*(g)(x, y) = g(\varphi_1(x, y)) = g(x) = \psi_1(g).$$

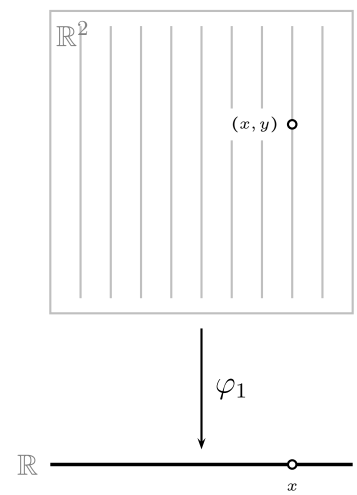
(b) We are looking for a map $\varphi_2 : \mathbb{R} \longrightarrow \mathbb{R}$ so that $\varphi_2^* = \psi_2$. Since φ_2 maps to $\mathbb{R}^2$ it is given by a pair of functions (g_1, g_2). From the remarks above, g_1 is the pullback of the coordinate function x, and g_2 is the pullback of the coordinate function y. If $\varphi_2^* = \psi_2$ then

$$\begin{aligned} g_1 &= \varphi_2^*(x) = \psi(x) = t, \text{ and} \\ g_2 &= \varphi_2^*(y) = \psi(y) = 0. \end{aligned}$$

Therefore the only possible choice for φ_2 is $\varphi_2(t) = (t, 0)$. The map φ_2^* is the homomorphism we are looking for. For any $f(x,y) \in \mathbb{R}[x,y]$,

$$\varphi_2^*(f)(t) = f(\varphi_2(t)) = f(t,0) = \psi_2(f).$$

(c) The map $\varphi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\varphi_1(x,y) = x$ is vertical projection of $\mathbb{R}^2$ onto $\mathbb{R}$.



The map $\varphi_2 : \mathbb{R} \longrightarrow \mathbb{R}^2$ given by $\varphi_2(t) = (t, 0)$ is inclusion of $\mathbb{R}$ into $\mathbb{R}^2$ as the x-axis.