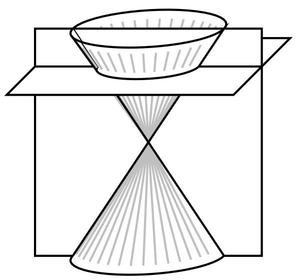
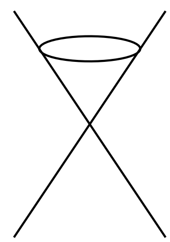
1. Draw pictures of the zero loci of the two equations $f_1 = xz - x$ and $f_2 = x^2 + y^2 - z^2$ in $\mathbb{A}^3$. Find their intersection and decompose it into irreducible components. Find the prime ideals in $k[x, y, z]$ associated to each component.

Solution. The zero locus of $f_1 = xz - x = x(z - 1)$ consists of the plane $x = 0$ and the plane $z = 1$. The zero locus of $f_2 = x^2 + y^2 - z^2$ is the cone from Homework 2, Question 1(a). Their intersection looks like this:



with components



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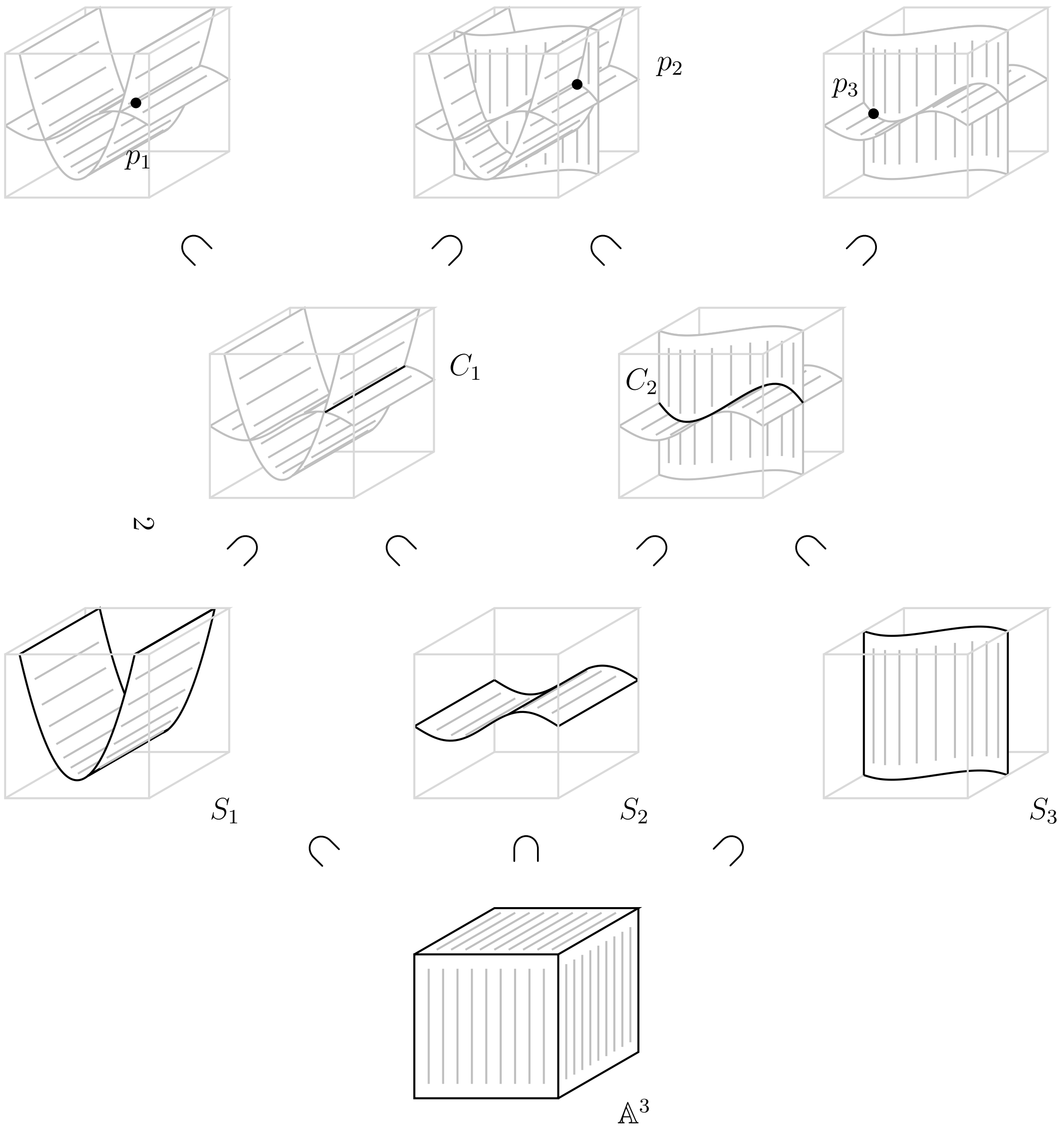
The components are the circle, with equations $z = 1$, $x^2 + y^2 = 1$, the line $x = 0$, $y = z$, and the line $x = 0$, $y = -z$. The respective ideals are $\langle z - 1, x^2 + y^2 - 1\rangle$, $\langle x, y - z\rangle$, and $\langle x, y + z\rangle$.

2. Draw pictures of the various kinds of irreducible subvarieties in $\mathbb{A}^3$, analogous to the one we drew in class for $\mathbb{A}^2$. Include a parallel diagram of corresponding prime ideals.

Solution. The picture appears on the next page. In that diagram, the maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$ correspond to points p_1, p_2, and p_3; the prime ideals Q_1 and Q_2 correspond to curves C_1 and C_2; prime ideals P_1, P_2, and P_3 correspond to surfaces S_1, S_2, and S_3; finally the prime ideal $\{0\}$ coresoponds to all of $\mathbb{A}^3$. The inclusion among the ideals corresponds to the inclusions among the irreducible subvarieties, but in the opposite direction.

We can read a few more facts off from this diagram. In the picture the curve C_2 is the intersection of the two surfaces S_2 and S_3; the corresponding relation among the prime ideals is $Q_2 = \sqrt{P_2 + P_3}$. However, C_1 is not the intersection of S_1 and S_2, that intersection has at least one other component. Therefore, the equations in P_1 and P_2 are not enough to generate Q_2 (not even up to radical), more equations are needed to get rid of the other component.



$$
\begin{tikzcd}
\mathfrak{m}_1 & & \mathfrak{m}_2 & & \mathfrak{m}_3 \\
& Q_1 \arrow[ul, phantom, sloped, "\subset"] \arrow[ur, phantom, sloped, "\supset"] & & Q_2 \arrow[ul, phantom, sloped, "\subset"] \arrow[ur, phantom, sloped, "\supset"] & \\
P_1 \arrow[ur, phantom, sloped, "\subset"] & & P_2 \arrow[ul, phantom, sloped, "\supset"] \arrow[ur, phantom, sloped, "\subset"] & & P_3 \arrow[ul, phantom, sloped, "\supset"] \\
& & \{0\} \arrow[ull, phantom, sloped, "\supset"] \arrow[u, phantom, sloped, "\subset"] \arrow[urr, phantom, sloped, "\subset"] & &
\end{tikzcd}
$$

3. Let $\mathbb{A}^4$ be thought of as the space of 2×2 matrices via the correspondence

$$(x_1, x_2, x_3, x_4) \leftrightarrow \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Let $U \subset \mathbb{A}^4$ be the subset consisting of 2×2 matrices with distinct eigenvalues. In this question we will show that U is a Zariski open set.

(a) For a quadratic polynomial $p(t) = at^2 + bt + c$, what is the algebraic condition on a, b, and c which determines when $p(t)$ has repeated roots?

(b) For a point $(x_1, x_2, x_3, x_4) \in \mathbb{A}^4$, write out the characteristic polynomial $p(t)$ of the corresponding matrix.

(c) Show that U is an open subset of $\mathbb{A}^4$ in the Zariski topology.

Solution.

(a) A quadratic polynomial $at^2 + bt + c$ (with $a \neq 0$) has repeated roots if and only if $b^2 - 4ac = 0$.

(b) $p(t) = \begin{vmatrix} t - x_1 & -x_2 \\ -x_3 & t - x_4 \end{vmatrix} = (t-x_1)(t-x_4)-(-x_2)(-x_3) = t^2-(x_1+x_4)t+(x_1x_4-x_2x_3)$.

(c) The matrices with distinct eigenvalues are those whose characteristic polynomials have no repeated roots. By parts (a) and (b), these are the matrices for which

$$(x_1 + x_4)^2 - 4(x_1x_4 - x_2x_3) = (x_1 - x_4)^2 - 4x_2x_3 \neq 0.$$

Thus the set U is the complement to the set of points where $(x_1 - x_4)^2 = 4x_2x_3$, and is therefore an open set.

4. In this question we will check the claim that finite unions of subvarieties are again subvarieties, and thus that the set of subvarieties of a given variety satisfies the axioms to be the closed subsets of a topological space.

By induction (or by repeating the argument) it is enough to check the case of the union of two subvarieties. Let X be an affine variety with ring of functions $R[X]$, and Z_1, Z_2 two closed subsets (i.e., subvarieties) of X with ideals J_1 and J_2.

(a) Show that $V(J_1 \cap J_2) = Z_1 \cup Z_2$.

(b) Give an example to show that an infinite union of closed subsets (in the Zariski topology) is not closed.

Solution.

(a) Let x be any point outside $Z_1 \cup Z_2$. Then since $x \notin Z_1$, there is an $f_1 \in J_1$ with $f_1(x) \neq 0$. Similarly, since $x \notin Z_2$ there is an $f_2 \in J_2$ with $f_2(x) \neq 0$. But then $f_1 f_2 \in J_1 \cap J_2$, and $(f_1 f_2)(x) = f_1(x) f_2(x) \neq 0$, so $x \notin V(J_1 \cap J_2)$. This shows the inclusion $V(J_1 \cap J_2) \subseteq Z_1 \cup Z_2$.

We now show the opposite inclusion. Suppose that $z \in Z_1 \cup Z_2$. Then z is either in Z_1 or Z_2, or both. For any $f \in J_1 \cap J_2$ we must therefore have $f(z) = 0$: if $z \in Z_1$ we use the fact that $f \in J_1$ to conclude that $f(z) = 0$, while if $z \in Z_2$ we use the fact that $f \in J_2$ to conclude the same thing. Thus points of $Z_1 \cup Z_2$ are in the zero loci of $J_1 \cap J_2$, or $Z_1 \cup Z_2 \subseteq V(J_1 \cap J_2)$.

The two inclusions show that $V(J_1 \cap J_2) = Z_1 \cup Z_2$.

(b) There are many possible examples. Here are two

(b1) Let $Z = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\} \subset \mathbb{A}^1$ (i.e, Z is the set of integer points). A point is closed in $\mathbb{A}^1$, so Z is the union of infinitely many closed subsets, but Z is not closed in the Zariski topology. Any polynomial $f \in k[x]$ which vanishes on Z has infinitely many roots, and so must be the zero polynomial.

(b2) Similarly, let $Z \subset \mathbb{A}^2$ be the union of the lines through the origin with slopes 0, 1, 2, 3, ... Each line is closed in the Zariski topology by the union is not: Any polynomial f vanishing on all the lines must be the zero polynomial. (One way to see this: intersect with any line of the form $y = c$ or vertical line $x = c$ [with $c \neq 0$ in both cases] to reduce to the example above and conclude that f is zero on that line. Since f is zero on each horizontal and vertical line away from the origin, f is the zero polynomial.)