1. Although we are talking about $\mathbb{P}^n$ over algebraically closed fields, and usually over $\mathbb{C}$, we can consider $\mathbb{P}^n$ over any field. If we consider $\mathbb{P}^n$ over a finite field, then $\mathbb{P}^n$ only has finitely many points with coordinates in the field. In this problem we will count the number of points in two different ways. Let p be a prime number.

(a) How many points does $\mathbb{A}^m$ have over $\mathbb{F}_p$?

(b) How many elements $\lambda \in \mathbb{F}_p$, $\lambda \neq 0$ are there?

(c) Considering $\mathbb{P}^n$ as $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ modulo the relation of scaling by elements of $\mathbb{F}_p^*$, how many points does $\mathbb{P}^n$ have over $\mathbb{F}_p$?

(d) We have seen that the complement of a standard $\mathbb{A}^n$ coordinate chart in $\mathbb{P}^n$ is a $\mathbb{P}^{n-1}$. Continuing in this way we get a decomposition of $\mathbb{P}^n$ into disjoint subsets:

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0.$$

Use this decomposition and part (a) to give a second formula for the number of points of $\mathbb{P}^n$ over $\mathbb{F}_p$.

(e) Check that your answers in (c) and (d) are the same.

(f) As a specific example, let $p = 2$. How many points does $\mathbb{P}^2$ have over $\mathbb{F}_2$? How many lines are there in $\mathbb{P}^2$ over $\mathbb{F}_2$? How many points are on each line?

Remarks. (1) We could also have considered the case that the field is $\mathbb{F}_q$, with $q = p^r$ a prime power. The formulas, with q taking the place of p, are the same. (2) If you have seen the card game "Spot It", you may want to also do the computations in (f) with $p = 7$.

Solution.

(a) The points of $\mathbb{A}^m$ over $\mathbb{F}_p$ are those points $(x_1, \ldots, x_m)$ with each $x_i \in \mathbb{F}_p$. Since there are p choices for each x_i, this is a total of $p \cdot p \cdot p \cdots p = p^m$ different points.

(b) There are $p - 1$ points of $\mathbb{F}_p$ which are not equal to 0.

(c) The (multiplicative) group $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ acts on $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ by the rule $\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n)$. We have defined $\mathbb{P}^n$ as the orbits under this action. Each orbit has exactly $p - 1$ elements, and since $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ has $p^{n+1} - 1$ elements, that means that there are $\frac{p^{n+1}-1}{p-1}$ orbits. I.e., $\mathbb{P}^n$ has $\frac{p^{n+1}-1}{p-1}$ points over $\mathbb{F}_p$.

(*d*) Alternately, using the decomposition

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0.$$

and part (a) we see that $\mathbb{P}^n$ has

$$p^n + p^{n-1} + p^{n-2} + \cdots + p^1 + p^0$$

points over $\mathbb{F}_p$.

(*e*) The answers in (c) and (d) are of course the same: the formula for summing a geometric series shows us that

$$p^n + p^{n-1} + p^{n-2} + \cdots + p^1 + 1 = \frac{p^{n+1} - 1}{p - 1}.$$

Remark. This counting problem is therefore a geometric incarnation of the geometric series.

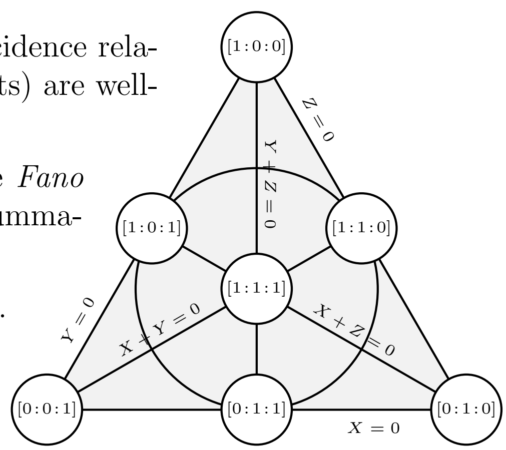
(*f*) Our formulas tell us that over $\mathbb{F}_2$, $\mathbb{P}^2$ has $\frac{2^3-1}{2-1} = 7$ points.

There are also 7 lines. Lines are given by equations $aX + bY + cZ = 0$, where $a, b, c \in \mathbb{F}_2$, not all are zero, and we only care about (a, b, c) up to scalar. Thus the lines in $\mathbb{P}^2$ are themselves parametrized by a $\mathbb{P}^2$, and so the number of lines is the same as the number of points, namely 7. Since each line is a $\mathbb{P}^1$, it has $\frac{2^2-1}{2-1} = 2 + 1 = 3$ points.

Remark. The example of $\mathbb{P}^2$ over $\mathbb{F}_2$, and its incidence relations (the data of which lines contain which points) are well-known example of a finite geometry.

In that setting, it often goes by the name of the *Fano Plane*, and the points and relations are usually summarized by the picture at right.

In the picture, the 7 points of $\mathbb{P}^2$ over $\mathbb{F}_2$ are shown. The lines of the triangle (and through the triangle) are the lines in $\mathbb{P}^2$ — each one contains 3 points.



The circle going through $[1 : 0 : 1]$, $[1 : 1 : 0]$, and $[0 : 1 : 1]$ is also represents a line, the line with equation $X + Y + Z = 0$.

In the picture, every two distinct points are contained on a unique line, every two distinct lines meet in a unique point, and every line contains three points, just as they are supposed to.

2. In $\mathbb{P}^n$, the zero locus of an equation of the form $a_0 Z_0 + a_1 Z_1 + \cdots + a_n Z_n$ is called a *hyperplane*. Given any k hyperplanes, $H_1, \ldots, H_k$ in $\mathbb{P}^n$ with $k \leqslant n$, show that their intersection $H_1 \cap H_2 \cap \cdots \cap H_k$ is nonempty.

Solution. Let hyperplane H_i be given by the equation $a_{i0}Z_0 + a_{i1}Z_1 + \cdots + a_{in}Z_n$. The $k \times (n+1)$ matrix

$$\begin{bmatrix} a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k0} & a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix}$$

has rank at most k, and so if $k \leqslant n$ there is a nonzero vector $(z_0, z_1, \ldots, z_n)$ in the kernel. The point $[Z_0 \colon Z_1 \colon \cdots \colon Z_n]$ is then a point of $\mathbb{P}^n$ on each of H_1,..., H_k, so that $H_1 \cap H_2 \cap \cdots \cap H_k \neq \varnothing$.

3. In this problem we will consider subvarieties of $\mathbb{P}^1$.

(*a*) Let X and Y be the homogeneous coordinates on $\mathbb{P}^1$, and let $x = [\alpha \colon \beta]$ be a point of $\mathbb{P}^1$. Show that the homogeneous polynomial $G = \beta X - \alpha Y$ has only a single zero, and that zero is at x.

(*b*) Let F be a homogeneous polynomial of degree d in X and Y. The zeros of F are a finite set of points. Show that the number of points, counted with multiplicity (i.e, counted according to the number of times each factor appears) is exactly d. As always, you should assume that the field k is algebraically closed.

Solution.

(*a*) The point $x = [\alpha \colon \beta] \in \mathbb{P}^1$ is clearly a zero of G since $G([\alpha \colon \beta]) = \beta(\alpha) - \alpha(\beta) = 0$. Now let $[u \colon v] \in \mathbb{P}^1$ be a zero of G. The condition that $G([u \colon v]) = 0$ is $u\beta - v\alpha = 0$. Perhaps the cleanest way to write this condition is as the condition that

$$\begin{vmatrix} u & v \\ \alpha & \beta \end{vmatrix} = 0.$$

But a 2×2 matrix has rank one if and only if one row is a multiple of the other. Since both rows are nonzero, we conclude that there is a $\lambda \in k$, $\lambda \neq 0$ so that $(u, v) = \lambda(\alpha, \beta)$. This means that the points $[u \colon v]$ and $[\alpha \colon \beta]$ are the same point of $\mathbb{P}^1$, so that $x = [u \colon v]$.

(*b*) Since k is algebraically closed, we can factor F as

$$F = \prod_{i=1}^{d} (\beta_i X - \alpha_i Y)$$

with $\alpha_i,\ \beta_i \in k$. (To see that this follows from k being algebraically closed, dehomogenize F to get polynomial f in one variable, factor f into a product of linear factors, and rehomogenize to get the above factorization of F.)

By part (a), the i-th factor has a single zero, the point $[\alpha_i\colon \beta_i] \in \mathbb{P}^1$. Thus F has d zeros, when counted with multiplicity.

4. We have seen that affine varieties are completely determined by their ring of global functions. In contrast, projective varieties are *not* determined by their ring of functions, in fact, they have very few global functions at all.

(a) Show that the only global algebraic functions on $\mathbb{P}^1$ are the constant functions. Do this by considering functions f_0 and f_1 in the standard coordinate charts U_0 and U_1, and looking at the conditions for these functions to agree on the intersection.

(b) Similarly show that the only global algebraic functions on $\mathbb{P}^2$ are the constant functions. You can do this by patching as in part (a), but perhaps a simpler argument is to use the fact that any two points $p, q \in \mathbb{P}^2$ are contained in a unique line, and that each line is a $\mathbb{P}^1$, and part (a).

After doing the question we see that the rings of global functions on $\mathbb{P}^1$ and $\mathbb{P}^2$ are the same, but $\mathbb{P}^1$ and $\mathbb{P}^2$ are certainly not isomorphic!

Solution.

(a) The standard open sets U_0 and U_1 are each isomorphic to $\mathbb{A}^1$. Let z be the coordinate on U_0 and w the coordinate on U_1. On the intersection the coordinates are related by the formula $z = \frac{1}{w}$. Writing out g_0 and g_1 as polynomials in z and w respectively, we have

$$g_0 = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_d z^d$$

and

$$g_1 = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots + b_e w^e$$

On the intersection we have $w = \frac{1}{z}$, so that on the intersection g_1 can also be written as

$$g_1 = b_0 + b_1\left(\frac{1}{z}\right) + b_2\left(\frac{1}{z}\right)^2 + b_3\left(\frac{1}{z}\right)^3 + \cdots + b_e\left(\frac{1}{z}\right)^e = \sum_{j=0}^{e} b_j z^{-j}$$

In order for g_0 and g_1 to be equal on the intersection, the coefficients of z must be equal, so that we must have $a_i = b_{-i}$ for all i. Since $a_i = 0$ when $i < 0$ and $b_j = 0$ when $j < 0$ this means that the only nonzero coefficients possible are a_0 and b_0, and these must be equal by the relation above. Thus, the only algebraic functions on $\mathbb{P}^1$ are the constant functions.

(b) Let f be an algebraic function on $\mathbb{P}^2$, and let p and q be any two distinct points in $\mathbb{P}^2$. As we have seen in class, there is a unique line ℓ containing both p and q. The line ℓ is a $\mathbb{P}^1$ (all lines in $\mathbb{P}^2$ are $\mathbb{P}^1$'s). Restricting f to ℓ, and using part (a), we get that f is a constant function on ℓ, and therefore that $f(p) = f(q)$.

We have therefore shown that given any two points p and q on $\mathbb{P}^2$, f takes the same values on those two points. It follows that f takes the same values on every point, and so f is constant.

Remark. The conclusion of this result is perhaps easier to see in the complex analytic world. Suppose that f is a complex analytic function on $\mathbb{P}^1$. Since f is continuous, and $\mathbb{P}^1$ compact, $|f|$ must obtain a maximum value at some point $p \in \mathbb{P}^1$. Restricting to an open neighbourhood U around p, we would then have a complex analytic function on U such that $|f|$ takes its maximum value on an interior point. By the maximum principle from complex analysis, that means that f must be constant on U. By analytic continuation we conclude that f is constant on all of $\mathbb{P}^1$. A similar argument (using the two-variable maximum principle) works for $\mathbb{P}^2$. The key difference between $\mathbb{P}^n$ and the affine case is that $\mathbb{P}^n$ is compact, and that puts strong restrictions on the global algebraic or holomorphic functions.