1. Here is an extremely simple example of a map between Riemann surfaces (aka "algebraic curves"). Fix an integer $n \geqslant 1$ and define a map $\varphi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ by the formula $[X\colon Y] \longrightarrow [X^n\colon Y^n]$.

(*a*) Check that φ is well-defined, that is (1) φ doesn't depend on the choice of representative we use for $[X\colon Y]$, and (2) no point of $\mathbb{P}^1$ is sent to $[0\colon 0]$ by these instructions.

In order to see that this is a map of Riemann surfaces, let us look in coordinate charts.

(*b*) Check that $\varphi^{-1}(U_0) = U_0$ and that $\varphi^{-1}(U_1) = U_1$, i.e, that φ maps the standard coordinate charts to the standard coordinate charts.

(*c*) In each of U_0 and U_1 write out (in the coordinates of each chart) what φ is doing. Is φ an algebraic map?

(*d*) Find all the ramification points of φ and their ramification degrees.

Solution.

(*a*) (1) Suppose that $p = [X : Y] \in \mathbb{P}^1$. For any $\lambda \in \mathbb{C}^*$, $[\lambda X : \lambda Y]$ represents the same point p. Since the coordinates of $\varphi([(\lambda X)^n : (\lambda Y)^n]) = [\lambda^n X : \lambda^n Y]$ are a λ^n times the coordinates of $\varphi([X : Y]) = [X^n : Y^n]$, they represent the same point in $\mathbb{P}^1$. Therefore the instructions for φ, do not depend on the homogeneous coordinates chosen to represent p.

(2) The only way that $\varphi([X : Y]) = [X^n : Y^n] = [0 : 0]$ is if $X^n = 0$ and $Y^n = 0$, which implies that $X = 0$ and $Y = 0$. Since $X = 0$, $Y = 0$ is not a point of $\mathbb{P}^1$, we conclude that there is no point $[X : Y] \in \mathbb{P}^1$ so that $\varphi([X : Y]) = [0 : 0]$, and so φ gives a well defined map of sets from $\mathbb{P}^1$ to $\mathbb{P}^1$.

(*b*) The coordinate chart U_0 is defined by the condition $X \neq 0$, so $[X : Y] \in \varphi^{-1}(U_0)$ exactly when $\varphi([X : Y]) = [X^n : Y^n]$ satisfies $X^n \neq 0$, which is the same condition as $X \neq 0$. In other words, $[X : Y] \in \varphi^{-1}(U_0)$ if and only if $[X : Y] \in U_0$, so $\varphi^{-1}(U_0) = U_0$.

Similarly, the coordinate chart U_1 is defined by the condition that $Y \neq 0$. Therefore $[X : Y] \in \varphi^{-1}(U_1)$ if and ony if $\varphi([X : Y]) = [X^n : Y^n]$ satisfies the condition $Y^n \neq 0$, which is the same as asking that $Y \neq 0$. Therefore, $[X : Y] \in \varphi^{-1}(U_1)$ if and only if $[X : Y] \in U_1$ and so $\varphi^{-1}(U_1) = U_1$.

(*c*) On U_0 the coordinate is $z = \frac{Y}{X}$. From the point of U_0, the map φ is the composite

$$z \leftrightarrow [1:z] \overset{\varphi}{\mapsto} [1^n : z^n] = [1 : z^n] \leftrightarrow z^n \in U_0.$$

That is, on U_0, φ is given by $\varphi(z) = z^n$.

Similarly, on U_1 with coordinate $w = \frac{X}{Y}$, φ is the composite

$$w \leftrightarrow [w:1] \overset{\varphi}{\mapsto} [w^n : 1^n] = [w^n : 1] \leftrightarrow w^n \in U_1,$$

so that $\varphi(w) = w^n$ on U_1.

From this description, φ is certainly an algebraic map!

(*d*) In chart U_0, the only ramification point is at 0 (corresponding to the point $p_0 = [1:0] \in \mathbb{P}^1$). From the coordinate description $z \mapsto z^n$, the ramification degree at p_0 is $k_{p_0} = n$.

In chart U_1, the only ramification point is at 0 (corresponding to the point $p_1 = [0:1] \in \mathbb{P}^1$). From the coordinate description $w \mapsto w^n$, the ramification degree at p_0 is $k_{p_1} = n$.

NOTE: The map φ is a map from $\mathbb{P}^1$ to $\mathbb{P}^1$ of degree n. As a check on our computations of the number and ramification degree of the ramification points of φ we should see that the Riemann-Hurwitz formula holds with this data. The computation is:

$$\begin{aligned} -2 = 2(0-1) &= 2(g(\mathbb{P}^1) - 1) \overset{\text{R–H}}{=\!=} n \cdot 2(g(\mathbb{P}^1) - 1) + \sum_p (k_p - 1) \\ &= n \cdot 2(0-1) + (n-1) + (n-1) = -2n + (2n-2) = -2. \end{aligned}$$

So, our computation seems reasonable – the Riemann-Hurwitz formula agrees that having two ramification points of degree n is compatible with a degree n cover from $\mathbb{P}^1$ to $\mathbb{P}^1$.

2. Use the Riemann-Hurwitz formula to find the genus of X, the genus of Y, or the number of ramification points, as required.

(*a*) $\pi : X \longrightarrow \mathbb{P}^1$ is a degree 3 cover, with two ramification points, both with ramification index $k_p = 3$. Find the genus of X.

(*b*) $\pi : X \longrightarrow \mathbb{P}^1$ is a degree 3 cover, with three ramification points, all with ramification index $k_p = 3$. Find the genus of X.

(c) $\pi : X \longrightarrow Y$ is a map of degree d, X has genus 1, and there are no ramification points. Find the genus of Y.

(d) X is of genus g, Y is of genus 1, the map $\pi : X \longrightarrow Y$ is of degree d, and all ramification points p in X are of index 2. Find the number of ramification points (the answer turns out, in this case, not to depend on the degree d).

Can you think of a map $X \longrightarrow \mathbb{P}^1$ satisfying the description in part (a)?

Solution.

(a) By the Riemann-Hurwitz formula,

$$2(g_X - 1) = 3 \cdot 2(0-1) + (3-1) + (3-1) = -6 + 4 = -2,$$

so that $g_X = 0$.

(b) By the Riemann-Hurwitz formula,

$$2(g_X - 1) = 3 \cdot 2(0-1) + (3-1) + (3-1) + (3-1) = -6 + 6 = 0,$$

so that $g_X = 1$.

(c) By the Riemann-Hurwitz formula,

$$0 = 2(1-1) = d \cdot 2(g_Y - 1) + 0 = 2(g_Y - 1),$$

so that $g_Y = 1$.

(d) Since $g_X = g$, $g_Y = 1$, and $k_p = 2$ for all ramification points, the Riemann-Hurwitz formula gives us

$$2g - 2 = 2 \cdot (g_X - 1) = d \cdot 2(1-1) + \sum_p (k_p - 1) = 0 + (\#\text{ ramification points}),$$

so that the number of ramification points is $2g - 2$.

The map in (a) is a map $\pi \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of degree 3, with two ramification points, each of ramification index $k_p = 3$. An example of such a map is the map considered in question 1 with $n = 3$, i.e,. the map $\varphi \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ given by $\varphi([X : Y]) = [X^3 : Y^3]$.

3. In this question we will complete the proof of the theorem describing the "global" picture of a non-constant map $\varphi\colon X \longrightarrow Y$ between Riemann surfaces. The key missing step of the theorem was this : to show that there exists a positive integer d, such that for any $q \in Y$, $\sum_{p\in\varphi^{-1}(q)} k_p = d$. Here the sum is over all p such that $\varphi(p) = q$, and k_p denotes the ramification index of φ at p.

To reduce notation somewhat, let us define the function $D\colon Y \longrightarrow \mathbb{N}$ by $D(q) = \sum_{p\in\varphi^{-1}(q)} k_p$. The goal of this problem is then to show that D is a constant function.

LEMMA : For each $q \in Y$ there is a small neighbourhood (= open set around) V of q such that D is constant on V.

First let us see how to prove the result using the lemma.

(*a*) Use the lemma to show that for each $d \in \mathbb{N}$ the set

$$D^{-1}(d) = \left\{ q \in Y \,\middle|\, D(q) = d \right\}$$

is open.

(*b*) Use (a) to show that for each $d \in \mathbb{N}$ the set $D^{-1}(d)$ is closed. (SUGGESTION: this is the same as showing that the complement is open.)

(*c*) Use (a)+(b) to show that for each $d \in \mathbb{N}$, $D^{-1}(d)$ is either Y or the empty set.

(*d*) Conclude that there is a unique $d \in \mathbb{N}$ such that $D^{-1}(d) = Y$, i.e., conclude that D is constant on Y.

We now work on proving the lemma.

Fix $q \in Y$, and suppose that $\varphi^{-1}(q) = \{p_1, p_2, \ldots, p_r\}$. From our local picture we know that there is an open set V around q, and open sets $U_1, \ldots, U_r$ around $p_1,\ldots, p_r$ such that $\varphi(U_i) \subset V$ for each $i = 1,\ldots, r$, and that on each U_i the map φ looks like $z_i \mapsto z_i^{k_{p_i}}$, where z_i is a local coordinate on U_i, and k_{p_i} the ramification index at p_i.

Given these U_i and V, for $q \in V$ let us split our function D into the sum of two functions. For $q' \in V$, by definition $D(q')$ is the sum over $p' \in \varphi^{-1}(q')$ of the ramification indices $k_{p'}$. We will split the sum into pieces according to whether p' is in $U_1 \cup U_2 \cup \cdots \cup U_r$ or outside it. Set $U = U_1 \cup U_2 \cup \cdots \cup U_r$ and define :

$$D_U(q') = \sum_{p'\in\varphi^{-1}(q')\cap U} k_{p'} \text{ and } D_U^c \sum_{p'\in\varphi^{-1}(q'),\, p\notin U} k_{p'},$$

so that $D(q') = D_U(q') + D_U^c(q')$. (The "$c$" is for "complement.)

Set $d = D(q) = k_{p_1} + k_{p_2} + \cdots + k_{p_r}$.

(e) Show that for q' sufficiently close to q, $D_U(q') = d$.

CLAIM: For q' sufficiently close to q, all points of $\varphi^{-1}(q')$ are in U. (This then shows that for those points $D_U^c(q') = 0$, and hence using $D = D_U + D_U^c$ and (e) that $D(q') = d$ for all points q' sufficiently close to q, thus proving the lemma.)

The negation of this claim is that there is a sequence of points q_1', q_2', ..., converging to q, and for each q_i' a point $p_i' \in \varphi^{-1}(q_i')$ which is outside of U. Since X is compact, such a sequence would have a limit point $\overline{p} \in X$.

(f) Explain why we would have $\varphi(\overline{p}) = q$.

(g) Explain why this means that $\overline{p} \in \{p_1, p_2, \ldots, p_r\}$.

(h) Explain why this means that some p_i' (in fact, infinitely many p_i') would have to be in U.

(i) Explain why this is a contradiction, thus establishing the claim, the lemma, and finally the theorem from class.

Solution.

(a) By definition a set S is open if for each points $q \in S$, there is an open set $V \subseteq S$ which contains q.

Fix $d \in \mathbb{N}$ and set $S = D^{-1}(d)$. If $q \in S$ then $D(q) = d$ (by definition of S). By the lemma, there is an open set V containing q so that D is constant on V, i.e., $D(q') = d$ for all $q' \in V$. Thus $V \subseteq S$, and so S is open.

(b) One way to prove that a set S is open is to prove that its complement is closed. Fix $d \in \mathbb{N}$ and set $S = D^{-1}(d)$. Let $S^c = Y \setminus S$ be the complement of S in Y. From the definition, a point $q \in S^c$ if and only if $q \notin S$, i.e., if and only if $D(q) \neq d$. From this we see that

$$S^c = \bigcup_{e \in \mathbb{N}, e \neq d} D^{-1}(e).$$

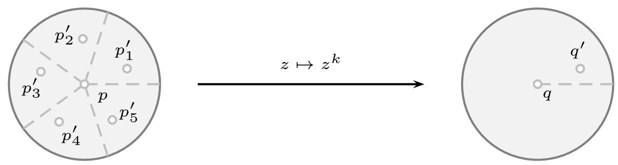
By part (a) each of the sets $D^{-1}(e)$ is open, and an arbitrary union of open sets is open, therefore S^c is open, and so S is closed.

(c) By (a) and (b), for each $d \in \mathbb{N}$ the set $D^{-1}(d)$ is both open and closed in Y. For a connected topological space (like Y), the only sets which are both open and closed are $\varnothing$ and Y. Thus, for each $d \in \mathbb{N}$, $D^{-1}(d)$ is either empty or all of Y.

(d) Let q be any point of Y, and $d = D(q)$. Then $q \in D^{-1}(d)$, so $D^{-1}(d) \neq \varnothing$. By part (c) this means that $D^{-1}(d) = Y$, i.e., that for all $q' \in Y$, $D(q') = d$, so that D is constant on Y.

(e) On each U_i we know that φ looks like the map $z_i \mapsto z_i^{k_{p_i}}$. As long as q' is close enough to q so that $q' \in \varphi(U_i)$, then $\varphi^{-1}(q') \cap U_i$ is the solutions to $z_i^{k_1} = w$, where w is the number corresponding to q' in the coordinate system on V.

Thus, once q' is close enough to q, $\varphi^{-1}(q') \cap U_i$ contains k_i points, since $z^{k_i} = w$ has exactly k_i solutions in $\mathbb{C}$ when $w \neq 0$ (i.e, when $q' \neq q$). Here is the usual picture of the map $z \to z^k$ illustrating this :



Thus, once q' is close enough to q to be inside all $\varphi(U_i)$ (i.e, $q' \in \bigcap_{i=1}^{r} \varphi(U_i)$), and when $q \neq q'$, then $\varphi^{-1}(q')$ have exactly k_{p_i} points in U_i, and so a total of $\sum_{i=1}^{r} k_{p_i} = d$ points in U.

However, each of those points is unramified, i.e., their ramification index is 1. (We saw this in class by a local description of what the "k" in the ramification index means.) Thus $D_U(q')$, which is the some of the ramification indices of the points of $\varphi^{-1}(q') \cap U$ is the sum of the number 1 over the d points in $\varphi^{-1}(q') \cap U$, and so $D_U(q') = d$. (If $q' = q$ then we already know that the points $\{p_1, \ldots, p_r\}$ of $\varphi^{-1}(q)$ all lie in U, and that their ramfication indices sum to d — that is how we defined d!)

Therefore, for all q' sufficiently close to q (including $q' = q$) $D_U(q') = d$.

(f) To make the notation easier, let us assume that we have already passed to a subsequence of the p_i' which converges to $\overline{p}$, i.e., that $\lim_{i\to\infty} p_i' = \overline{p}$.

We know that the q_i' converge to q, and that $\varphi(p_i') = q_i'$ for each i. Since φ is a continuous map we therefore have

$$\varphi(\overline{p}) = \varphi\left(\lim_{i\to\infty} p_i'\right) = \lim_{i\to\infty} \varphi(p_i') = \lim_{i\to\infty} q_i' = q.$$

(g) Since $\varphi(\overline{p}) = q$, $\overline{p} \in \varphi^{-1}(q) = \{p_1, p_2, \ldots, p_r\}$.

(h) By (g), $\overline{p} = p_j$ for some j, $1 \leqslant j \leqslant r$, and by definition U_j is an open set around p_j. Since the sequence $\{p_i'\}$ is converging to $\overline{p} = p_j$, then there is some N so that for all $i \geqslant N$, $p_i' \in U_j$.

(i) The p_i' were chosen so that no p_i' lies in any U_j. The conclusion above is therefore a contradiction, and so there is no such sequence q_i' converging to q, establishing the claim. (And therefore the lemma, and then the theorem!)