1. In this problem we will prove that $\sqrt{\langle x^2(x+1), y\rangle} = \langle x(x+1), y\rangle$.

(a) Explain why we have the containment $\langle x(x+1), y\rangle \subseteq \sqrt{\langle x^2(x+1), y\rangle}$.

From part (a), in order to show equality it is enough to show the reverse containment. Let f be any element of $\sqrt{\langle x^2(x+1), y\rangle}$.

(b) Explain why we know that there is an $n \geq 1$ and polynomials $h_1, h_2 \in k[x,y]$ such that

(b1)
$$f^n = x^2(x+1)h_1 + yh_2.$$

(c) Let $\psi\colon k[x,y] \longrightarrow k[x]$ be the ring homomorphism given by setting $y=0$, and set $\overline{f} = \psi(f)$. Looking at the image of (b1) under ψ, and using unique factorization in the ring $k[x]$, explain why we know that there is a polynomial $h_3 \in k[x]$ so that
$$\overline{f} = x(x+1)h_3.$$

(d) Using part (c), explain why we know that there is a polynomial $h_4 \in k[x,y]$ so that $f - x(x+1)h_4$ is in the kernel of ψ.

(e) What is the kernel of ψ?

(f) Complete the problem by showing that $f \in \langle x(x+1), y\rangle$.

2. In this problem we will explore other questions about the radical.

(a) Let A be any ring, $I \subset A$ and ideal, and $f \in I$. Suppose that $f = f_1^{e_1} f_2^{e_2} \cdots f_r^{e_r}$ for some $f_1, \ldots, f_r \in A$, and some $e_1, \ldots e_r \geq 1$. Show that $f_1 f_2 \cdots f_r \in \sqrt{I}$.

(b) Let $I \subset \mathbb{Z}$ be an ideal. We know that every ideal in $\mathbb{Z}$ is generated by a single element, so $I = \langle n\rangle$ for some $n \in \mathbb{Z}$. Assume that $n \neq 0$ (i.e, $I \neq (0)$) and let $n = p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization of n. Show that $\sqrt{I} = \langle p_1 p_2 \cdots p_r\rangle$.

(c) Let J_1 and J_2 be ideals. Show that $J_1 \cap J_2$ is also an ideal.

(d) Let I_1 and I_2 be radical ideals. Show that $I_1 \cap I_2$ is also a radical ideal.

[Math 813 only] (e) For any $f \in k[x_1, \ldots, x_n]$ let $f = f_1^{e_1} \cdots f_r^{e_r}$ be its factorization into irreducibles, and define $\mathrm{Rad}(f)$ by the formula $\mathrm{Rad}(f) = f_1 f_2 \cdots f_r$. Show that if I is a principal ideal, $I = \langle f\rangle$, then $\sqrt{I} = \langle \mathrm{Rad}(f)\rangle$.

[Math 813 only] (f) Give an example of an ideal $I = \langle g_1, g_2\rangle \subset k[x,y]$ such that $\sqrt{I} \neq \langle \mathrm{Rad}(g_1), \mathrm{Rad}(g_2)\rangle$. (ONE POSSIBILITY: An ideal with this property has already appeared in class, but you can make up your own.)

3. Let $\mathfrak{m} \subset \mathbb{C}[x, y, z]$ be the maximal ideal $\mathfrak{m} = \langle x - 3, y - 4, z - 5 \rangle$. Which of the following ideals are contained in $\mathfrak{m}$? And how do you know?

(a) $I_1 = \langle x^2 + y^2 - z^2 \rangle$.

(b) $I_2 = \langle z^2 - 2xy \rangle$.

(c) $I_3 = \langle y^2 - x^2 - x - y, xyz - 3z^2 + 5x \rangle$.

(d) $I_4 = \langle x^2 + y^2 + z^2 - xy - xz - yz, 7yz + 4xz - 8z^2 \rangle$.

[Math 813 only] 4. In order that maximal ideals are in one-to-one correspondence with points, we needed the condition that k be algebraically closed. In this problem we will see in a simple example what happens if k is not algebraically closed: Maximal ideals are in one-to-one correspondence with $\mathrm{Gal}(\overline{k}/k)$ orbits of points.

[Math 813 only] (a) Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ be the Galois group of $\mathbb{C} = \overline{\mathbb{R}}$ over $\mathbb{R}$. Classify the orbits of G on $\mathbb{C}$.

[Math 813 only] (b) Classify the maximal ideals of $\mathbb{R}[x]$.

[Math 813 only] (c) Show that the maximal ideals of $\mathbb{R}[x]$ are in one-to-one correspondence with the orbits of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}$.