Math 413/813 — Homework Assignment 6

DUE DATE: FEB. 26, 2019

1. Draw pictures (in $\mathbb{R}^3$) of the zero loci of the two equations $f_1 = xz - x$ and $f_2 = x^2+y^2-z^2$ in $\mathbb{A}^3$. Find their intersection and decompose it into irreducible components. Find the prime ideals in $\mathbb{C}[x,y,z]$ associated to each component.

2. Draw pictures of various kinds of irreducible subvarieties in $\mathbb{A}^3$, analogous to the one we drew in class for $\mathbb{A}^2$. Be sure to have a subvariety of each possible dimension. Include a parallel diagram of corresponding prime ideals.

(The ideals to not have to be specific prime ideals, they can be nonspecific prime ideals with names like P, Q, etc. The point is to again practice the geometry/algebra correspondence by visualizing how the lattice of irreducible subvarieties corresponds to the lattice of prime ideals. Here "lattice" means "set with a partial order".)

3. Let $\mathbb{A}^4$ be thought of as the space of 2×2 matrices via the correspondence

$$(x_1,x_2,x_3,x_4) \longleftrightarrow \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Let $U \subset \mathbb{A}^4$ be the subset consisting of 2×2 matrices with distinct eigenvalues. In this question we will show that U is a Zariski open set.

(a) For a quadratic polynomial $p(t) = at^2+bt+c$, what is the algebraic condition on a, b, and c which determines when $p(t)$ has repeated roots?

(b) For a point $(x_1,x_2,x_3,x_4) \in \mathbb{A}^4$, write out the characteristic polynomial $p(t)$ of the corresponding matrix.

(c) Show that U is an open subset of $\mathbb{A}^4$ in the Zariski topology.

4. In this question we will check the claim that finite unions of subvarieties are again subvarieties, and thus that the set of subvarieties of a given variety satisfies the axioms to be the closed subsets of a topological space.

By induction (or by repeating the argument) it is enough to check the case of the union of two subvarieties. Let X be an affine variety with ring of functions $R[X]$, and Z_1, Z_2 two closed subsets (i.e., subvarieties) of X with ideals J_1 and J_2.

(a) Show that $V(J_1 \cap J_2) = Z_1 \cup Z_2$.

(b) Give an example to show that an infinite union of closed subsets (in the Zariski topology) is not closed.