Math 413/813 Homework Assignment 7

DUE DATE: MAR. 5, 2019

1. Let $X = \mathbb{A}^n$ and for $2 \leqslant s \leqslant n$ let V_s be the open subset of X which is the complement of the linear space $x_1 = x_2 = x_3 = \cdots = x_s = 0$. Compute (analogously to the computation for $\mathbb{A}^2$ and $s = 2$) the ring of functions $\mathcal{O}_X(V_s)$. (You can make your life easier in the case $s > 2$ by appealing to your answer for $s = 2$.)

2. Let X be the affine variety described by the equation $xy - z^2 = 0$ in $\mathbb{A}^3$, and let $U \subset X$ be the complement of $(0, 0, 0) \in X$. In this problem we will compute $\mathcal{O}_X(U)$ and see that it is equal to $R[X]$.

The variety X is covered by the principal open sets U_x and U_y, with coordinate rings $k[x, y, z, 1/x]/\langle xy - z^2\rangle \cong k[x, 1/x, z]$ and $k[x, y, z, 1/y]/\langle xy - z^2\rangle \cong k[y, 1/y, z]$ respectively. Any function $g_1 \in R[U_x]$ can be written as a finite sum $g_1 = \sum a_{ij}x^iz^j$ and any function $g_2 \in R[U_y]$ can be written as a finite sum $g_2 = \sum b_{k\ell}y^kz^\ell$.

(a) What ranges of indices are valid in the expressions for g_1 and g_2 above?

We want to look at pairs (g_1, g_2) which agree on $U_x \cap U_y$. The expressions for g_1 and g_2 above are with respect to different variables. To compare them we need to write them in terms of the same variables.

(b) Use the relation $y = \frac{z^2}{x}$ (valid on U_x, and hence also on $U_x \cap U_y$) to write g_2 in terms of the variables x and z.

(c) In order for g_1 to be equal to g_2, what must be the relation between the a_{ij} and the $b_{k\ell}$?

(d) Considering the restrictions on the indices from part (a), your formula from (c) will imply additional restrictions on i and j. What are they?

(e) For each i and j satisfying the conditions above, show that there is a monomial $x^py^qz^r$ which is equal to x^iz^j on U_x.

(f) Explain why this means that the restriction homomorphism $R[X] \longrightarrow \mathcal{O}_X(U)$ is surjective.

3. Given a ring A and an element $f \in A$ we have been looking at the ring $A[1/f]$ obtained by adjoining the additional element $1/f$ to A (and of course using ring operations to get more elements). More precisely the ring $A[1/f]$ is the ring $A[y]/\langle yf - 1\rangle$. There is

a natural ring homomorphism $A \longrightarrow A[1/f]$, which is the composite of the ring homomorphsms $A \hookrightarrow A[y] \longrightarrow A[y]/\langle yf-1\rangle$. We have seen in class that this homomorphism is not always injective. For instance, if $h \in A$ is an element so that $h \cdot f^n = 0$ in A for some $n \geqslant 1$, then in $A[1/f]$ we compute that $h = (h \cdot f^n) \cdot \frac{1}{f^n} = 0 \cdot \frac{1}{f^n} = 0$.

The purpose of this question is to prove the converse direction: An element $h \in A$ is in the kernel of the map $A \longrightarrow A[1/f]$ only if there is an $n \geqslant 1$ such that $h \cdot f^n = 0$ in A.

Suppose that h is such an element. This means that the image of h under the inclusion $A \hookrightarrow A[y]$ must be in the ideal $\langle yf-1\rangle$ in $A[y]$. Therefore there is a polynomial $g \in A[y]$ such that $h = g(yf-1)$. Since $g \in A[y]$ we can write g as $g = g_0 + g_1 y + g_2 y^2 + \cdots + g_n y^n$ with each $g_j \in A$.

(a) Expand $g \cdot (yf-1)$ as a polynomial in y.

(b) As a polynomial in y, h has degree 0. Since we have $h = g \cdot (yf-1)$, the coefficients of powers of y on both sides of the equality must be the same. Comparing coefficients, write down all the relations you obtain.

(c) Show that $h \cdot f^{n+1} = 0$.

4. A topological space X is called *quasi-compact* if whenever $\{U_i\}_{i\in S}$ are a family of open subsets such that $\cup_{i\in S} U_i = X$ then there are a finite number of U_i's which actually cover X. (The term *compact* is reserved for Hausdorff topological spaces with this finite subcover property. A topological space with the finite subcover property alone is called quasi-compact.) In this question we will show that affine varieties are quasi-compact.

(a) Show that if U_{f_i}, $i \in S$ is a family of principal open subsets which cover an affine variety X, then there is a finite number which cover X. (Suggestion: Think about what the condition "the U_{f_i} cover X" means about the complement.)

(b) Given an arbitrary cover of X by open sets $\{U_j\}$, $j \in S$, use the fact that the principal open subsets are a basis for the Zariski topology and part (a) to show that a finite number of the U_j are sufficient to cover X.

5. In class we have seen that if X is an affine variety and U_f a principal open subset, then U_f is an affine variety. Perhaps every open subset is an affine variety? The purpose of this question is to show that the answer to this is no. Let $U = \mathbb{A}^2 \setminus \{(0,0)\}$. Recall that we have computed that $\mathcal{O}_{\mathbb{A}^2}(U) = k[x,y] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$.

(a) Let $\varphi\colon U \hookrightarrow \mathbb{A}^2$ be the inclusion map. Compute the ring homomorphism φ^*.

(b) Maps between affine varieties are completely determined by the pullback maps. If U were an affine variety, explain why φ would have to be an isomorphism.

(c) Show that U is not an affine variety.