DUE DATE: MAR. 12, 2019

1. Although we are talking about $\mathbb{P}^n$ over algebraically closed fields, and usually over $\mathbb{C}$, we can consider $\mathbb{P}^n$ over any field. If we consider $\mathbb{P}^n$ over a finite field, then $\mathbb{P}^n$ only has finitely many points with coordinates in the field. In this problem we will count the number of points in two different ways. Let p be a prime number.

(*a*) How many points does $\mathbb{A}^m$ have over $\mathbb{F}_p$?

(*b*) How many elements $\lambda \in \mathbb{F}_p$, $\lambda \neq 0$ are there?

(*c*) Considering $\mathbb{P}^n$ as $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ modulo the relation of scaling by elements of $\mathbb{F}_p^*$, how many points does $\mathbb{P}^n$ have over $\mathbb{F}_p$?

(*d*) We have seen that the complement of a standard $\mathbb{A}^n$ coordinate chart in $\mathbb{P}^n$ is a $\mathbb{P}^{n-1}$. Continuing in this way we get a decomposition of $\mathbb{P}^n$ into disjoint subsets:

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{A}^{n-2} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0.$$

Use this decomposition and part (*a*) to give a second formula for the number of points of $\mathbb{P}^n$ over $\mathbb{F}_p$.

(*e*) Check that your answers in (*c*) and (*d*) are the same.

(*f*) As a specific example, let $p = 2$. How many points does $\mathbb{P}^2$ have over $\mathbb{F}_2$? How many lines are there in $\mathbb{P}^2$ over $\mathbb{F}_2$? How many points are on each line?

REMARKS. (1) We could also have considered the case that the field is $\mathbb{F}_q$, with $q = p^r$ a prime power. The formulas, with q taking the place of p, are the same. (2) If you have seen the card game "Spot It", you may want to also do the computations in (f) with $p = 7$.

2. In $\mathbb{P}^n$, the zero locus of an equation of the form $a_0Z_0 + a_1Z_1 + \cdots + a_nZ_n$ is called a *hyperplane*. Given any k hyperplanes, $H_1, \ldots, H_k$ in $\mathbb{P}^n$ with $k \leqslant n$, show that their intersection $H_1 \cap H_2 \cap \cdots \cap H_k$ is nonempty.

3. In this problem we will consider subvarieties of $\mathbb{P}^1$.

(*a*) Let X and Y be the homogeneous coordinates on $\mathbb{P}^1$, and let $p = [\alpha\colon \beta]$ be a point of $\mathbb{P}^1$. Show that the homogeneous polynomial $G = \beta X - \alpha Y$ has only a single zero, and that zero is at p.

(b) Let F be a homogeneous polynomial of degree d in X and Y. The zeros of F are a finite set of points. Show that the number of points, counted with multiplicity (i.e, counted according to the number of times each factor appears) is exactly d. As always, you should assume that the field k is algebraically closed.

4. We have seen that affine varieties are completely determined by their ring of global functions. In contrast, projective varieties are *not* determined by their ring of functions, in fact, they have very few global functions at all.

(a) Show that the only global algebraic functions on $\mathbb{P}^1$ are the constant functions. Do this by considering functions f_0 and f_1 in the standard coordinate charts U_0 and U_1, and looking at the conditions for these functions to agree on the intersection.

(b) Similarly show that the only global algebraic functions on $\mathbb{P}^2$ are the constant functions. You can do this by patching as in part (a), but perhaps a simpler argument is to use the fact that any two points $p, q \in \mathbb{P}^2$ are contained in a unique line, and that each line is a $\mathbb{P}^1$, and part (a).

After doing the question we see that the rings of global functions on $\mathbb{P}^1$ and $\mathbb{P}^2$ are the same, but $\mathbb{P}^1$ and $\mathbb{P}^2$ are certainly not isomorphic!

NOTE: In (a) the idea is to do a "patching" computation like we have previously done to determine the ring of functions on an open subset of an affine variety, althought this time the set is all of $\mathbb{P}^1$. We know two affine open subsets, U_0 and U_1 which cover $\mathbb{P}^1$, and we know how they are glued together on their common intersection, and that is all we need to compare a function f_0 on U_0 restricted to $U_0 \cap U_1$ and a function f_1 on U_1 restricted to $U_0 \cap U_1$