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1. In this problem we will use right exactness to compute some tensor products.

(a) Let m and n be positive integers, and set $d = \gcd(m, n)$. Prove that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$.

Now fix a field k and set $\mathrm{A} = k[x, y]$; $\mathrm{M}_1 = \mathrm{A}/(x)$; $\mathrm{M}_2 = \mathrm{A}/(x-y)$; and $\mathrm{M}_3 = \mathrm{A}/(x-1)$. All of M_i are A-modules.

Compute

(b) $\mathrm{M}_1 \otimes_{\mathrm{A}} \mathrm{M}_2$.

(c) $\mathrm{M}_1 \otimes_{\mathrm{A}} \mathrm{M}_3$.

(d) $\mathrm{M}_2 \otimes_{\mathrm{A}} \mathrm{M}_3$.

The modules in (a), (b), and (c) are finite dimensional vector spaces over k. When you compute them, try and describe them in the simplest way possible, and also give their dimensions as vector spaces over k.

2. In class we showed that if $M \longrightarrow N$ is a surjection of A-modules with kernel $P \subseteq M$, then $\mathrm{T}^\bullet(M) \longrightarrow \mathrm{T}^\bullet(N)$ is surjective, and has kernel the two-sided ideal generated by $P \subseteq \mathrm{T}^1(M) = M$.

Let $\mathrm{A} = \mathbb{Z}$, let n be a positive integer, and set $\mathrm{N} = \mathbb{Z}/n\mathbb{Z}$. Use this theorem to compute $\mathrm{T}^\bullet(\mathrm{N})$. In particular, explicitly describe the $\mathbb{Z}$-module $\mathrm{T}^p(\mathrm{N})$ for each $p \geqslant 1$, and describe the multiplication map $\mathrm{T}^p(\mathrm{N}) \otimes \mathrm{T}^q(\mathrm{N}) \longrightarrow \mathrm{T}^{p+q}(\mathrm{N})$.

3. We know that the functor $\mathrm{T}^\bullet$ takes a surjection to a surjection; in this question we see that $\mathrm{T}^\bullet$ does not in general take injections to injections.

(a) Let $\mathrm{A} = \mathbb{Z}$, $\mathrm{N} = \mathbb{Z}/4\mathbb{Z}$, M the submodule of N generated by $\overline{2} \in \mathrm{N}$, and $\varphi : \mathrm{M} \hookrightarrow \mathrm{N}$ the inclusion map. Show that $\mathrm{T}^\bullet(\varphi) : \mathrm{T}^\bullet(\mathrm{M}) \longrightarrow \mathrm{T}^\bullet(\mathrm{N})$ is not an injection.

Let us return to the case that A is a general commutative ring. An A-submodule M of N is called a *direct factor* if there exists $j\colon \mathrm{N} \longrightarrow \mathrm{M}$ such that $j \circ i = \mathrm{Id}_{\mathrm{M}}$, where $i\colon \mathrm{M} \hookrightarrow \mathrm{N}$ is the inclusion map.

(b) Show that if M is a direct factor of N then $\mathrm{T}^\bullet(i)\colon \mathrm{T}^\bullet(\mathrm{M}) \longrightarrow \mathrm{T}^\bullet(\mathrm{N})$ is injective. (SUGGESTION: $\mathrm{T}^\bullet$ is a functor!)