DUE DATE: APR. 1, 2024

1. Suppose that V is an n-dimensional vector space over a field k, and that $\varphi : V \longrightarrow V$ is a linear transformation with eigenvalues $\lambda_1, \ldots, \lambda_n$. Find the eigenvalues of

(a) $\mathrm{Sym}^2(\varphi) : \mathrm{Sym}^2(V) \longrightarrow \mathrm{Sym}^2(V)$, and

(b) $\Lambda^2(\varphi) : \Lambda^2 V \longrightarrow \Lambda^2 V$.

SUGGESTION: As in the case of the tensor product, pick a good basis $e_1, \ldots, e_r$ for V over k, and use our results on bases for $\mathrm{Sym}^2(V)$ and $\Lambda^2 V$ and the formulae for $\mathrm{Sym}^2(\varphi)$ and $\Lambda^2(\varphi)$ to find the eigenvalues.

Use this to prove the following formulas (where Tr is the trace)

(c) $\mathrm{Tr}(\mathrm{Sym}^2(\varphi)) = \dfrac{1}{2}\left(\mathrm{Tr}(\varphi)^2 + \mathrm{Tr}(\varphi^2)\right)$, and

(d) $\mathrm{Tr}(\Lambda^2\varphi) = \dfrac{1}{2}\left(\mathrm{Tr}(\varphi)^2 - \mathrm{Tr}(\varphi^2)\right)$.

2. Let V be a two-dimensional vector space over a field k, with basis e_1, e_2. Let $\varphi\colon V \longrightarrow V$ be a linear transformation, with matrix (in the basis e_1, e_2)

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The vector space $\Lambda^2 V$ is one-dimensional, spanned by $e_1 \wedge e_2$. Thus, the map $\Lambda^2(\varphi)\colon \Lambda^2 V \longrightarrow \Lambda^2 V$ is a map from a one-dimensional vector space to itself, and so corresponds to multiplication by a number.

(a) Compute $\Lambda^2(\varphi)(e_1 \wedge e_2)$, and find that number.

(b) What does "functoriality of Λ^2" correspond to in this case? (I.e., given another linear transformation $\psi\colon V \longrightarrow V$, what commonly known fact does the equation $\Lambda^2(\psi \circ \varphi) = \Lambda^2(\psi) \circ \Lambda^2(\varphi)$ correspond to?)

In general, if V is an n-dimensional vector space, then any linear transformation $\varphi\colon V \longrightarrow V$ gives a map $\Lambda^n(\varphi)$, a map from the one-dimensional vector space $\Lambda^n V$ to itself. The map $\Lambda^n(\varphi)$ is therefore multiplication by a number, called the determinant of φ. If M is a matrix representing φ, we also use $\det(M)$ for that number.

3. Let V be an n-dimensional vector space over a field k, with basis $e_1, \ldots, e_n$.

For any subset $S = \{i_1, i_2, \ldots, i_p\}$ of $\{1, 2, \ldots, n\}$ of size p let e_S denote the element $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$ of $\wedge^q V$, where $i_1 < i_2 < \cdots < i_p$ are taken in increasing order. For instance, if $n = 4$, and $S = \{1, 2, 4\}$ then $e_S = e_1 \wedge e_2 \wedge e_4$.

For any subset S of $\{1, 2, \ldots, n\}$ let S' be the complementary subset (i.e, $S' = \{1, 2, \ldots, n\} \setminus S$), and define the sign $\xi_{S,S'}$, which will be either $+1$ or -1, by the formula

$$e_S \wedge e_{S'} = \xi_{S,S'}\, e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_n.$$

Let $\varphi : V \longrightarrow V$ be a linear transformation from V to V and M the matrix for φ with respect to the basis $e_1, \ldots, e_n$.

Finally, for any subsets C (the columns) and R (the rows) of $\{1, 2, \ldots, n\}$ of the same size p, let $M_{C,R}$ be the $p \times p$ submatrix of M constructed from the columns in C and the rows in R.

For any subset C of $\{1, \ldots, n\}$ of size p you may assume (or prove if you like) the formula

$$\Lambda^p \varphi(e_C) = \sum_R \det(M_{C,R}) e_R \in \Lambda^p V$$

where the sum is over all subsets R of $\{1, \ldots, n\}$ of size p.

(a) Using this, the definition of the determinant in terms of the alternating product from the previous page, and the fact that $\Lambda^\bullet \varphi$ is an algebra homomorphism from $\Lambda^\bullet V$ to $\Lambda^\bullet V$ prove the

Laplace expansion formula: For any subset C of $\{1, \ldots, n\}$ of size p, $1 \leqslant p < n$,

$$\det(M) = \xi_{C,C'} \sum_R \xi_{R,R'} \det(M_{C,R}) \det(M_{C',R'})$$

where the sum is again over all subsets R of $\{1, \ldots, n\}$ of size p.

In the special case that $p = 1$, so that C consists of a single number this is the expansion formula for the determinant "down a column", but the actual formula proved by Laplace is more general.

(b) Carry out a sample computation in the case that $n = 4$, $C = \{1, 3\}$, and the matrix M is

$$M = \begin{bmatrix} 1 & 3 & 5 & 2 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

Suggestion : The fact that "$\Lambda^\bullet$ is an algebra homomorphism" is saying, for instance, that $\Lambda^n(e_C \wedge e_{C'}) = \Lambda^p(e_C) \wedge \Lambda^{n-p}(e_{C'})$ for any subset $C \subseteq \{1, \ldots, n\}$ of size p.