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1. Let Σ_n be the symmetric group on $\{1, \ldots, n\}$, and let $V = \mathbb{C}^n$ with basis $e_1, \ldots, e_n$. We make V into a representation of Σ_n by the formula $\sigma \cdot e_i = e_{\sigma(i)}$ for all $i = 1, \ldots, n$, and $\sigma \in \Sigma_n$. This representation is called the *permutation representation* of Σ_n. Let

$$W = \left\{ (x_1, \ldots, x_n) \in V \;\middle|\; \sum_{i=1}^{n} x_i = 0 \right\}.$$

The subspace W is stable under the action of Σ_n and so is itself a representation of Σ_n, called the *standard representation of* Σ_n. In this question we will show that W is an irreducible representation of Σ_n.

(a) Find a basis for W.

(b) Let $w \in W$, $w \neq 0$ be any element. By acting on w with elements of Σ_n and taking linear combinations, show that you can obtain every one of the basis elements from (a).

(c) Prove that W is an irreducible representation of Σ_n.

2. We have seen in characteristic zero (or more generally characteristic p if $p \nmid |G|$) that every subrepresentation of a representation of a finite group has a G-stable complement, and so the representation splits as a direct sum. In this question we will see that this is not necessarily true in characteristic p when p divides the order of the group.

Fix a prime p and let $G = \{e, \sigma, \sigma^2, \ldots, \sigma^{p-1}\}$ be the cyclic group of order p with generator σ. Let V be the two dimensional vector space over $\mathbb{F}_p$ and define a representation ρ by $\rho(\sigma^j) = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}$ for $j = 0, \ldots, p-1$.

(a) Check that ρ really is a representation, that is, that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$.

(b) The subspace $W = \left\{ (x, 0) \in V \;\middle|\; x \in \mathbb{F}_p \right\}$ is stable under the action of G, and so defines a subrepresentation of G. Prove that there is no subspace $W' \subset V$ stable under the action of G and complementary to W.

3. Let G be a finite group, and V an irreducible representation of G over a field k. From Schur's lemma we know that every nonzero element of $\mathrm{Hom}_G(V,V)$ is an isomorphism, and that $\dim_k \mathrm{Hom}_G(V,V) = 1$ if k is algebraically closed. In this problem we will see that this second property can fail if k is not algebraically closed.

Let $k = \mathbb{Q}$ and $G = \{e, \sigma, \sigma^2, \sigma^3\}$ be the cyclic group of order 4 with generator σ. Let $V = \mathbb{Q}^2$ and make V into a representation of G by $\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We first check that V is an irreducible representation of G (over $\mathbb{Q}$).

(a) If V were a reducible representation of G over $\mathbb{Q}$, explain why V would have to split over $\mathbb{Q}$ as the direct sum of two one-dimensional representations.

(b) Explain why each of these one-dimensional subspaces would have to be eigenspaces of $\rho(\sigma)$.

(c) Compute the eigenvalues of $\rho(\sigma)$ and show that V is an irreducible representation of G.

We now compute $\mathrm{Hom}_G(V,V)$, those endomorphisms of V which commute with the action of G.

(d) Show that if $\varphi \in \mathrm{Hom}(V,V)$ satisfies $\varphi \circ \rho(\sigma) = \rho(\sigma) \circ \varphi$, then it satisfies $\varphi \circ \rho(g) = \rho(g) \circ \varphi$ for all $g \in G$. I.e., to check that $\varphi \in \mathrm{Hom}_G(V,V)$, we only need to check that $\varphi \circ \rho(\sigma) = \rho(\sigma) \circ \varphi$.

By (d) we have

$$\mathrm{Hom_G(V,V)} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ a, b, c, d \in \mathbb{Q} \right\}.$$

(e) Compute $\mathrm{Hom}_G(V,V)$ and find its dimension over $\mathbb{Q}$.

(f) By the first part of Schur's lemma, $\mathrm{Hom}_G(V,V)$ is a *division ring*, a (possibly non-commutative) ring such that every nonzero element is invertible. (Thus if $\mathrm{Hom}_G(V,V)$ is commutative, it is a field.) Which division ring did you find in (e)?