Math 894 Homework Assignment 6

DUE DATE: MAR. 1, 2018

1. Suppose that V is an n-dimensional vector space over a field k, and that $\varphi : V \longrightarrow V$ is a linear transformation with eigenvalues $\lambda_1, \ldots, \lambda_n$. Find the eigenvalues of

(a) $\mathrm{Sym}^2(\varphi) : \mathrm{Sym}^2(V) \longrightarrow \mathrm{Sym}^2(V)$, and

(b) $\Lambda^2(\varphi) : \Lambda^2 V \longrightarrow \Lambda^2 V$.

SUGGESTION: As in the case of the tensor product, pick a good basis $e_1, \ldots, e_r$ for V over k, and use our results on bases for $\mathrm{Sym}^2(V)$ and $\Lambda^2 V$ and the formulae for $\mathrm{Sym}^2(\varphi)$ and $\Lambda^2(\varphi)$ to find the eigenvalues.

Use this to prove the following formulas (where Tr is the trace)

(c) $\mathrm{Tr}(\mathrm{Sym}^2(\varphi)) = \frac{1}{2}\left(\mathrm{Tr}(\varphi)^2 + \mathrm{Tr}(\varphi^2)\right)$, and

(d) $\mathrm{Tr}(\Lambda^2\varphi) = \frac{1}{2}\left(\mathrm{Tr}(\varphi)^2 - \mathrm{Tr}(\varphi^2)\right)$.

2. Let V be an n-dimensional vector space over a field k, with basis $e_1, \ldots, e_n$.

For any subset $S = \{i_1, i_2, \ldots, i_p\}$ of $\{1, 2, \ldots, n\}$ of size p let e_S denote the element $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$ of $\wedge^q V$, where $i_1 < i_2 < \cdots < i_p$ are taken in increasing order. For instance, if $n = 4$, and $S = \{1, 2, 4\}$ then $e_S = e_1 \wedge e_2 \wedge e_4$.

For any subset S of $\{1, 2, \ldots, n\}$ let S' be the complementary subset (i.e, $S' = \{1, 2, \ldots, n\} \setminus S$), and define the sign $\xi_{S,S'}$, which will be either $+1$ or -1, by the formula

$$e_S \wedge e_{S'} = \xi_{S,S'}\, e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_n.$$

Let $\varphi : V \longrightarrow V$ be a linear transformation from V to V and M the matrix for φ with respect to the basis $e_1, \ldots, e_n$.

Finally, for any subsets C (the columns) and R (the rows) of $\{1, 2, \ldots, n\}$ of the same size p, let $M_{C,R}$ be the $p \times p$ submatrix of M constructed from the columns in C and the rows in R.

For any subset C of $\{1, \ldots, n\}$ of size p you may assume (or prove if you like) the formula

$$\Lambda^p\varphi(e_C) = \sum_R \det(M_{C,R}) e_R \in \Lambda^p V$$

where the sum is over all subsets R of $\{1,\ldots,n\}$ of size p.

Using this, the definition of the determinant in terms of the alternating product, and the fact that $\Lambda^\bullet\varphi$ is an algebra homomorphism from $\Lambda^\bullet V$ to $\Lambda^\bullet V$ prove the

Laplace expansion formula: For any subset C of $\{1,\ldots,n\}$ of size p, $1 \leqslant p < n$,

$$\det(M) = \xi_{C,C'} \sum_R \xi_{R,R'} \det(M_{C,R}) \det(M_{C',R'})$$

where the sum is again over all subsets R of $\{1,\ldots,n\}$ of size p.

In the special case that $p = 1$, so that C consists of a single number this is the expansion formula for the determinant "down a column", but the actual formula proved by Laplace is more general.

Carry out a sample computation in the case that $n = 4$, $C = \{1,3\}$, and the matrix M is

$$M = \begin{bmatrix} 1 & 3 & 5 & 2 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

3. Suppose that A is a commutative ring, and that N is a finitely generated A module, generated by $n_1,\ldots,n_q$ in N.

(a) Show that $\Lambda^p N = 0$ for all $p > q$.

Now suppose that $\mathfrak{m}$ is a maximal ideal of A with quotient field $k = A/\mathfrak{m}$, and let N and M be finitely generated A-modules.

(b) If $\dim_k(N \otimes_A A/\mathfrak{m}) = s$, show that N cannot be generated by fewer than s elements (it may require many many more elements, but you just have to show that it can't be done in fewer than s).

(c) If $\dim_k(M \otimes_A A/\mathfrak{m}) = r$, show that $\mathrm{Sym}^p(M)$ cannot be generated by fewer than $\binom{p+r-1}{p}$ elements.

This shows (among other things) that if $M \neq 0$ then $\mathrm{Sym}^p(M) \neq 0$ for all $p > 0$, in contrast with the result for the alternating product in part (a).

Suggestion: For part (c) you will probably want combine (b) with the result on "change of rings" for the symmetric powers, a consequence of the change of rings theorem for the tensor algebra. Here is how that theorem works. For any A-module M, and ring homomorphism $\varphi\colon A \longrightarrow B$, we can form the A-algebra $\mathrm{Sym}^\bullet_A(M)$, and then tensor to get a B-algebra $B \otimes_A \mathrm{Sym}_A^\bullet(M)$. On the other hand, we could have first promoted M to the B-module $B \otimes_A M$, and then computed the symmetric algebra over B : $\mathrm{Sym}_B^\bullet(B \otimes_A M)$. The result (for the symmetric algebra) is this : There is a canonical graded isomorphism of B-algebras :

$$\mathrm{Sym}_B^\bullet(B \otimes_A M) \xrightarrow{\sim} B \otimes_A \mathrm{Sym}_A^\bullet(M).$$

(There is a similar result for the alternating algebra.)

PUZZLE: If you would like some more multilinear challenges, you may enjoy the following (optional) puzzle.

You have a friend who does not like "alternating maps" but is happy with determinants.

(a) Suppose that V is a three-dimensonal vector space over a field k. You tell your friend: "Any time you need an alternating map $V \times V \longrightarrow k$, just pick a 2×3 matrix A, and define a map $V \times V \longrightarrow k$ by $(v, w) \mapsto \det_2(Av, Aw)$." (Where $\det_2(a, b)$ means the determinant of the 2×2 matrix with columns a and b).

First check that this procedure really produces an alternating map from $V \times V$ to k, and then prove all possible alternating maps from $V \times V$ to k may be realized this way.

(b) Heartened by your success in part (a), you decide that this trick will always work. That is, if V is an n-dimensional vector space, then every alternating map

$$\underbrace{V \times V \times \cdots \times V}_{p \text{ copies}} \longrightarrow k$$

can be realized by picking an appropriate $p \times n$ matrix A, and sending $(v_1, \ldots, v_p)$ to $\det_p(Av_1, Av_2, \ldots, Av_p)$. (Where $\det_p$ is the determinant of the $p \times p$ matrix with the columns listed).

Are you correct? If so, give an argument. If not, classify those n and p where this procedure really does produce all alternating maps from V^p to k. To simplify arguments, you should assume that k is algebraically closed.