Math 894 — Homework Assignment 8

DUE DATE: MAR. 15, 2018

Throughout the assignment we work over an algebraically closed field k of characteristic zero.

Character is destiny.
— Heraclitus

1. Let G be a finite group with irreducible representations $W_1, \ldots, W_h$, and let V be a finite dimensional representation of G. As usual we can decompose V as a direct sum of irreducibles :

$$V = m_1 W_1 \oplus m_2 W_2 \oplus \cdots \oplus m_h W_h.$$

This decomposition is not usually unique. For instance, imagine that V^G is three-dimensional, so that the trivial representation "appears three times". In decomposing V as a direct sum of irreducibles we must make a choice how to decompose V^G into three one-dimensional subspaces, and there are many ways to do that.

However, there an aspect of uniqueness to the decomposition. For any irreducible W_i, the subspace spanned by all the components isomorphic to W_i *is* uniquely determined (e.g., the subspace V^G, the sum of all the trivial representations, is uniquely determined). These components are called the *isotypic components*. In this problem we will prove that the isotypic components are unique by finding the G-homomorphism projectors onto them.

Let $d_1, \ldots, d_h$ be the dimensions of the irreducible representations, and $\chi_1, \ldots, \chi_h$ their characters. For each i, $i = 1, \ldots, h$, set $f_i = d_i \chi_i$. Also recall the construction of the G-endomorphisms φ_f for any class function f.

(a) For each i, show that φ_{f_i} acts as the identity on W_i, and as the zero map on W_j when $j \neq i$.

(b) For any representation V show that the G-endomorphism φ_{f_i} is a projection map with image the subspace of V spanned by the irreducibles of type W_i.

REMARK. When W_i is the trivial representation, $d_i = 1$ and $\chi_i = 1$, so that $f_i = 1$ and $\chi_{f_i} = \chi_1 = \mathrm{Avg_G}$ is the G-averaging operator. I.e., the operators above generalize the averaging operator which, as we have seen, is projection onto the fixed subspace V^G.

2. Let G be a finite group, X a finite set with G-action, (V, ρ) the corresponding permutation representation of G, and χ its character.

(a) Show that the average number of fixed points an element $g \in G$ has on X is equal to the number of times that V contains the trivial representation. (SUGGESTION: write down a formula for the average number of fixed points and reinterpret this as a pairing $\langle \cdot, \cdot \rangle$ between characters.)

(b) Show that the number of times that V contains the trivial representation is also equal to the number of orbits of G on X. (SUGGESTION: Find a bijection between orbits and a set of basis vectors of V^G.)

REMARK. (a) and (b) together show that the average number of fixed points of G acting on X is equal to the number of orbits, i.e., this gives a representation-theory proof of a result known as "Burnside's lemma".

(c) Suppose that $G = \Sigma_n$ the symmetric group, $X = \{1, 2, \ldots, n\}$, that G acts on X by the usual permutations, and let V be the associated permutation representation. We have seen that V splits as a direct sum of the trivial representation and an irreducible $(n-1)$-dimensional representation, called the *standard representation*. Prove that $\chi_{\text{std}}(g) = \#\{\text{fixed points of } g\} - 1$ for all $g \in G$.

3. Let V_{Std} be the standard representation of Σ_n as defined **H7** Q1 (or Q2(c) above). In this problem we will show that $\Lambda^k V_{\text{Std}}$ is irreducible for $k = 1, \ldots, n-1$.

(a) Suppose that for a representation W we know that $\langle \chi_W, \chi_W \rangle = 2$. Show that W is the direct sum of two irreducible representations, each appearing with multiplicity one.

(b) Suppose that V is a vector space and that $V = V_1 \oplus V_2$ where V_1 is a one-dimensional vector space. Show that $\Lambda^k V = (V_1 \otimes \Lambda^{k-1} V_2) \oplus \Lambda^k V_2$. (Don't forget results we already know from class.)

(c) Let V be the permutation representation as in 2(c) and compute $\langle \chi_{\Lambda^k V}, \chi_{\Lambda^k V} \rangle$.

(d) Use the results above to show that $\Lambda^k V_{\text{Std}}$ is irreducible.

Part (c) will require some non-trivial combinatorics.