Math 894 — Homework Assignment 9

DUE DATE: MAR. 22, 2018

1. Work out the character table for Σ_5. (**H7** Q1 and **H8** Q3 should give you a lot to start with, and you can deduce the missing entries as we have done for Σ_4.)

2. Let V_{Std} be the standard representation of Σ_5. Decompose

(a) $\text{Sym}^2(V_{\text{Std}})$, (NOTE: Don't forget **H6** Q1(a).)

(b) $V_{\text{Std}} \otimes V_{\text{Std}}$, and

(c) $V_{\text{Std}} \otimes V_{\text{Std}} \otimes V_{\text{Std}}$

into irreducible representations. (This means you should say how many times each irreducible appears, and not that you should provide an explicit decomposition into those irreducibles.)

3. As in **H8** Q1, let G be a finite group, X a finite set with G-action, (V, ρ) the corresponding permutation representation of G, and χ_V its character.

(a) Show that $\chi_V(g) = \chi_V(g^{-1})$ for all $g \in G$ (this will be used in (b) below).

An action of G on a set X is said to be *transitive* if for any x_1, $x_2 \in X$, there is at least one $g \in G$ with $g \cdot x_1 = x_2$. The action is said to be *doubly transitive* if it is transitive, and if for any $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq y_1$ and $x_2 \neq y_2$ there is at least one $g \in G$ with $g \cdot x_1 = x_2$ and $g \cdot y_1 = y_2$.

Suppose that X is a set with at least two elements, and that G acts transitively on X. Since this means that there is only one orbit, by **H8** Q2(b) this means that we can write the permutation representation V as $V = V_{\text{Triv}} \oplus V'$, where V_{Triv} is the trivial representation.

(b) Under these conditions show that the following are equivalent :

(b1) The action of G on X is doubly transitive

(b2) The action of G on $X \times X$ has only two orbits : the diagonal and its complement.

(b3) $\langle \chi_V^2, \chi_{\text{Triv}} \rangle = 2$.

(b4) The representation V' is irreducible.

In (b2) G acts on $X \times X$ by the 'diagonal action', namely $g \cdot (x, y) = (g \cdot x, g \cdot y)$.