1. Let A be a ring and

$$0 \xrightarrow{i} M_1 \longrightarrow M_2 \xrightarrow{\pi} M_3 \longrightarrow 0$$

a short exact sequence of A-modules. For any A-module N, show that the following two sequences (obtained by applying $\mathrm{Hom}_A(N,-)$ and $\mathrm{Hom}_A(-,N)$ respectively) are exact.

(a)
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(N,M_1) & \xrightarrow{i\circ} & \mathrm{Hom}(N,M_2) & \xrightarrow{\pi\circ} & \mathrm{Hom}(N,M_3) \\ & & \tau & \longmapsto & i\circ\tau; \quad \psi & \longmapsto & \pi\circ\psi \end{array}$$

(b)
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(M_3,N) & \xrightarrow{\circ\pi} & \mathrm{Hom}(M_2,N) & \xrightarrow{\circ i} & \mathrm{Hom}(M_1,N) \\ & & \psi & \longmapsto & \psi\circ\pi; \quad \tau & \longmapsto & \tau\circ i \end{array}$$

2. In this problem we will study a few aspects of flat and projective modules.

Recall that an exact sequence $0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{\pi} M_3 \longrightarrow 0$ is said to split if $M_2 \cong M_1 \oplus M_3$ and if, under the isomorphism, i corresponds to the natural injection $M_1 \hookrightarrow M_1 \oplus M_3$ and π the natural projection $M_1 \oplus M_3 \longrightarrow M_3$. One criterion for a sequence to split (which may have appeared in Algebra I) is the existence of a map $\psi\colon M_3 \longrightarrow M_2$ such that $\psi \circ \pi = \mathrm{Id}_{M_3}$. Equivalently, the sequence splits if there is a map $\tau\colon M_2 \longrightarrow M_1$ such that $\tau \circ i = \mathrm{Id}_{M_1}$.

(a) Let P be a projective A-module. Show that every exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow P \longrightarrow 0$ splits.

(b) Similarly, let I be an injective A-module. Show that every exact sequence $0 \longrightarrow I \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ splits.

In class we have seen that every A-module M is the quotient of a free module (we discussed this when discussing presentations).

(c) Let P be a projective module. Show that there is a free module F and a decomposition $F = P \oplus P'$ (where P' is some other A-module). I.e., that "every projective module is the direct summand of a free module".

(d) Show conversely that if F is a free module, and if $F \cong P_1 \oplus P_2$, then each of P_1 and P_2 are projective modules. (SUGGESTION: Prove that P_i has the defining property of a projective module, deducing this from the fact that a free module has this property, and the splitting as a direct sum.)

Thus, by combining (c) and (d) we have an alternate characterization of projective modules : an A-module P is projective if and only if it is the direct summand of a free module.

(e) Suppose that an A-module N decomposes as a direct sum : $N = N_1 \oplus N_2$. Show that N is a flat A-module if and only if both N_1 and N_2 are flat A-modules.

(f) Show that every projective A-module is also a flat A-module.

Remark. One can also show that every finitely generated flat module is projective, so for finitely generated A-modules flat and projective are equivalent. However, there are (non-finitely generated) flat modules which are not projective. For example, the ring $\mathbb{Z}[\frac{1}{2}]$ is a flat module over $\mathbb{Z}$, but not projective. (To see that $\mathbb{Z}[\frac{1}{2}]$ is not projective, note that every element of $\mathbb{Z}[\frac{1}{2}]$ is infinitely 2-divisible : given any $x \in \mathbb{Z}[\frac{1}{2}]$, and any $m \geqslant 0$ there exists $y \in \mathbb{Z}[\frac{1}{2}]$ such that $2^m y = x$. If $\mathbb{Z}[\frac{1}{2}]$ were a projective module, then by (c) it would be a direct summand of a free module. But no element of a free $\mathbb{Z}$-module is infinitely 2-divisible.)

3. Let $A = k$ be a field.

(a) Show that every A-module is projective.

(b) Show that every A-module is injective.

Now suppose that A is a domain (and commutative ring) for which every projective module is injective. We will now show that A is a field. Do do this, it suffices to show that for every $a \in A$, $a \neq 0$, there is a $b \in A$ such that $a \cdot b = 1$.

(c) Given $a \in A$, let $\varphi_a \colon A \longrightarrow A$ by the multiplication by a map : $\varphi_a(x) = ax$. Since A is a free module, it is also projective and so (by hypothesis) an injective module. Use the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi_a"] \arrow[d, "\mathrm{Id}_A"'] & A \\ & A & \end{tikzcd}$$

to show that there is a $b \in A$ with $ab = 1$.