1. This problem concerns details of the snake lemma omitted in class. The diagram for the snake lemma is shown on the back page.

(a) Explain how i' and π' induce the maps i'_{Q} and π'_{Q}. (I.e., what are these maps?)

(b) Show that the snake sequence

$$\mathrm{K}_1 \xrightarrow{i_{\mathrm{K}}} \mathrm{K}_2 \xrightarrow{\pi_{\mathrm{K}}} \mathrm{K}_3 \xrightarrow{\delta} \mathrm{Q}_1 \xrightarrow{i'_{\mathrm{Q}}} \mathrm{Q}_2 \xrightarrow{\pi'_{\mathrm{Q}}} \mathrm{Q}_3$$

is exact at Q_2.

(c) Show that the snake sequence is exact at Q_1.

2. An <u>injective resolution</u> of a module M is a cochain resolution $\mathrm{M} \longrightarrow \mathrm{I}^\bullet$ with each I^i an injective module. Suppose that we have an exact sequence $0 \longrightarrow \mathrm{M}_1 \longrightarrow \mathrm{M}_2 \longrightarrow \mathrm{M}_3 \longrightarrow 0$ of A-modules, and that $\mathrm{M}_1 \longrightarrow \mathrm{I}^\bullet$ and $\mathrm{M}_3 \longrightarrow \mathrm{K}^\bullet$ are injective resolutions of M_1 and M_3 respectively. Following the general outline of the proof of the horseshoe lemma for projective resolutions, prove a horseshoe lemma for injective resolutions. That is, show that it is possible to fill in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{M}_1 \arrow[r] \arrow[d, "d_0"] & \mathrm{M}_2 \arrow[r] & \mathrm{M}_3 \arrow[r] \arrow[d, "d''_0"] & 0 \\
 & \mathrm{I}^0 \arrow[d, "d_1"] & & \mathrm{K}^0 \arrow[d, "d''_1"] & \\
 & \mathrm{I}^1 \arrow[d, "d_2"] & & \mathrm{K}^1 \arrow[d, "d''_2"] & \\
 & \vdots & & \vdots &
\end{tikzcd}$$

with an injective resolution $\mathrm{M}_2 \longrightarrow \mathrm{J}^\bullet$ with $\mathrm{J}^i = \mathrm{I}^i \oplus \mathrm{K}^i$ for all $i \geqslant 0$ such that the natural inclusion and quotient maps give a short exact sequence of chain complexes $\mathrm{I}^\bullet \longrightarrow \mathrm{J}^\bullet \longrightarrow \mathrm{K}^\bullet$. (SUGGESTION : Injective modules are the "opposites" of projective modules, that is, their defining property is like that of projective modules but with the arrows reversed. Try "reversing all the arrows" in the proof of the horseshoe lemma for projective resolutions to get a proof for injective resolutions. The proof for projective resolutions is included in this homework assignment, starting on page 3.)

3. Let $\mathrm{C}_\bullet$ be the chain complex

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \cdots$$

where the differential maps are all "multiplication by 2".

(a) Show that $\mathrm{H}_i(\mathrm{C}_\bullet) = 0$ for all i, i.e, show that $\mathrm{C}_\bullet$ is an exact sequence.

The zero morphism $0_{\mathrm{C}_\bullet}\colon \mathrm{C}_\bullet \longrightarrow \mathrm{C}_\bullet$ and the identity map $\mathrm{Id}_{\mathrm{C}_\bullet}\colon \mathrm{C}_\bullet \longrightarrow \mathrm{C}_\bullet$ are maps of complexes. By (a) they induce the same map on homology (on the zero module, the zero map and the identity map are the same).

(b) Show that there does not exist a homotopy between these two maps. (Suggestion: Write out the equation that the homotopy would have to satisfy, and see if it can be satisfied when $1 \in \mathbb{Z}/4\mathbb{Z}$ is plugged into the equation.)

This gives an example of maps of chain complexes which induce the same maps on homology without being homotopic.

<u>Diagram for the Snake Lemma</u>

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, dashed] & \mathrm{K}_1 \arrow[r, "i_{\mathrm{K}}"] \arrow[d] & \mathrm{K}_2 \arrow[r, "\pi_{\mathrm{K}}"] \arrow[d] & \mathrm{K}_3 \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
0 \arrow[r, dashed] & \mathrm{M}_1 \arrow[r, "i"] \arrow[d, "f_1"] & \mathrm{M}_2 \arrow[r, "\pi"] \arrow[d, "f_2"] & \mathrm{M}_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & \mathrm{M}'_1 \arrow[r, "i'"] \arrow[d] & \mathrm{M}'_2 \arrow[r, "\pi'"] \arrow[d] & \mathrm{M}'_3 \arrow[r, dashed] \arrow[d] & 0 \\
 & \mathrm{Q}_1 \arrow[r, "i'_{\mathrm{Q}}"] \arrow[d] & \mathrm{Q}_2 \arrow[r, "\pi'_{\mathrm{Q}}"] \arrow[d] & \mathrm{Q}_3 \arrow[r, dashed] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Lemma ("Horseshoe Lemma") : Let $0 \longrightarrow \mathrm{M}_1 \longrightarrow \mathrm{M}_2 \longrightarrow \mathrm{M}_3 \longrightarrow 0$ be a short exact sequence of A-modules, and $\mathrm{P}_\bullet \longrightarrow \mathrm{M}_1$ and $\mathrm{R}_\bullet \longrightarrow \mathrm{M}_3$ projective resolutions of M_1 and M_3 respectively. Then there is a projective resolution $\mathrm{Q}_\bullet \longrightarrow \mathrm{M}_2$ with $\mathrm{Q}_i = \mathrm{P}_i \oplus \mathrm{R}_i$ for all $i \geqslant 0$, such that the natural inclusion and quotient maps give a short exact sequence of chain complexes $\mathrm{P}_\bullet \longrightarrow \mathrm{Q}_\bullet \longrightarrow \mathrm{R}_\bullet$.

Remark. It is important in the application that Q_i splits as a direct sum — that is, we want more than just a projective resolution $\mathrm{Q}_\bullet$ fitting in an exact sequence.

Proof. By induction on i. We will do the cases $i = 0, 1$ (from which the general case is clear).

$\boxed{i=0}$: We start with the following diagram, where δ_0 and δ_0'' are surjective

$$\begin{tikzcd}
 & \mathrm{P}_0 \arrow[d, "\delta_0"] & & \mathrm{R}_0 \arrow[d, "\delta_0''"] \arrow[dl, dashed, "\exists\, \varphi_0"'] & \\
0 \arrow[r] & \mathrm{M}_1 \arrow[r, "i"] & \mathrm{M}_2 \arrow[r, "\pi"] & \mathrm{M}_3 \arrow[r] & 0
\end{tikzcd}$$

Since π is surjective, and R_0 projective, by the universal property of projective modules there exists a morphism $\varphi\colon \mathrm{R}_0 \longrightarrow \mathrm{M}_2$ such that $\delta_0'' = \pi \circ \varphi_0$. We define a morphism $\delta_0'\colon \mathrm{P}_0 \oplus \mathrm{R}_0 \longrightarrow \mathrm{M}_2$ by $\delta_0' = (i \circ \delta_0) \oplus \varphi$, and thus obtain a commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{P}_0 \arrow[r] \arrow[d, "\delta_0"] & \mathrm{P}_0 \oplus \mathrm{R}_0 \arrow[r] \arrow[d, "\delta_0'"] & \mathrm{R}_0 \arrow[r] \arrow[d, "\delta_0''"] & 0 \\
0 \arrow[r] & \mathrm{M}_1 \arrow[r, "i"] & \mathrm{M}_2 \arrow[r, "\pi"] & \mathrm{M}_3 \arrow[r] & 0
\end{tikzcd}$$

Let $\mathrm{K}_0 = \mathrm{Ker}(\delta_0)$, $\mathrm{K}_0' = \mathrm{Ker}(\delta_0')$, and $\mathrm{K}_0'' = \mathrm{Ker}(\delta_0'')$. Applying the snake lemma to the diagram above and using the surjectivity of δ_0 and δ_0'' we get the long exact sequence

$$0 \longrightarrow \mathrm{K}_0 \longrightarrow \mathrm{K}_0' \longrightarrow \mathrm{K}_0'' \longrightarrow 0 \longrightarrow \mathrm{Coker}(\delta_0') \longrightarrow 0$$

from which we conclude that δ_0' is surjective, and that the kernels of δ_0, δ_0', and δ_0'' form a short exact sequence.

$\boxed{i=1}$: By hypothesis (i.e., the existence of the resolution) we have surjective maps $\mathrm{P}_1 \longrightarrow \mathrm{K}_0$ and $\mathrm{R}_1 \longrightarrow \mathrm{K}_0''$, i.e., we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{P}_1 \arrow[r] \arrow[d] & \mathrm{P}_1 \oplus \mathrm{R}_1 \arrow[r] \arrow[d] & \mathrm{R}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{K}_0 \arrow[r] & \mathrm{K}_0' \arrow[r] & \mathrm{K}_0'' \arrow[r] & 0
\end{tikzcd}$$

Repeating the construction from step $i = 0$, we obtain a surjective map $\mathrm{P}_1 \oplus \mathrm{R}_1 \longrightarrow \mathrm{K}_0'$ which makes the diagram commute and the top row into a split exact sequence. Splicing the vertical maps with the inclusions $\mathrm{K}_0 \hookrightarrow \mathrm{P}_0$, $\mathrm{K}_0' \hookrightarrow \mathrm{P}_0 \oplus \mathrm{R}_0$, and $\mathrm{K}_0'' \hookrightarrow \mathrm{R}_0$, gives the next step of the resolution.

Passing to the kernels of the vertical maps (and again using the Snake Lemma to see that the kernels form an exact sequence) we continue inductively for $i \geqslant 2$. $\square$