1. This problem concerns details of the snake lemma omitted in class. The diagram for the snake lemma is shown on the back page.

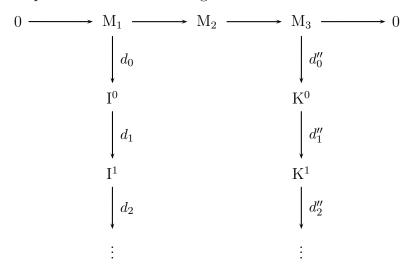
- (a) Explain how i' and π' induce the maps i'_Q and π'_Q . (I.e., what are these maps?)
- (b) Show that the snake sequence

 $K_1 \xrightarrow{i_K} K_2 \xrightarrow{\pi_K} K_3 \xrightarrow{\delta} Q_1 \xrightarrow{i'_Q} Q_2 \xrightarrow{\pi'_Q} Q_3$

is exact at Q_2 .

(c) Show that the snake sequence is exact at Q_1 .

2. An <u>injective resolution</u> of a module M is a cochain resolution $M \longrightarrow I^{\bullet}$ with each I^{i} an injective module. Suppose that we have an exact sequence $0 \longrightarrow M_{1} \longrightarrow M_{2} \longrightarrow$ $M_{3} \longrightarrow 0$ of A-modules, and that $M_{1} \longrightarrow I^{\bullet}$ and $M_{3} \longrightarrow K^{\bullet}$ are injective resolutions of M_{1} and M_{3} respectively. Following the general outline of the proof of the horseshoe lemma for projective resolutions, prove a horseshoe lemma for injective resolutions. That is, show that it is possible to fill in the diagram



with an injective resolution $M_2 \longrightarrow J^{\bullet}$ with $J^i = I^i \oplus K^i$ for all $i \ge 0$ such that the natural inclusion and quotient maps give a short exact sequence of chain complexes $I^{\bullet} \longrightarrow J^{\bullet} \longrightarrow K^{\bullet}$. (SUGGESTION : Injective modules are the "opposites" of projective modules, that is, their defining property is like that of projective modules but with the arrows reversed. Try "reversing all the arrows" in the proof of the horseshoe lemma for projective resolutions to get a proof for injective resolutions. The proof for projective resolutions is included in this homework assignment, starting on page 3.)

3. Let C_{\bullet} be the chain complex

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \cdots$$

where the differential maps are all "multiplication by 2".

(a) Show that $H_i(C_{\bullet}) = 0$ for all *i*, i.e., show that C_{\bullet} is an exact sequence.

The zero morphism $0_{C_{\bullet}}: C_{\bullet} \longrightarrow C_{\bullet}$ and the identity map $Id_{C_{\bullet}}: C_{\bullet} \longrightarrow C_{\bullet}$ are maps of complexes. By (a) they induce the same map on homology (on the zero module, the zero map and the identity map are the same).

(b) Show that there does not exist a homotopy between these two maps. (SUGGES-TION: Write out the equation that the homotopy would have to satisfy, and see if it can be satisfied when $1 \in \mathbb{Z}/4\mathbb{Z}$ is plugged into the equation.)

This gives an example of maps of chain complexes which induce the same maps on homology without being homotopic.

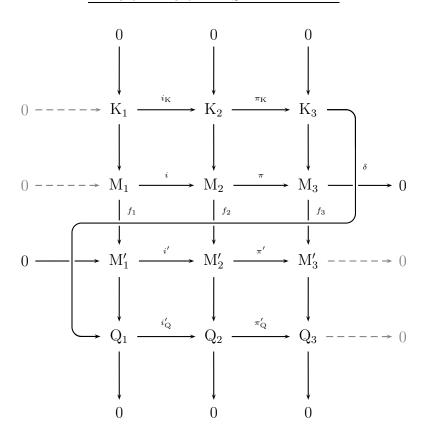


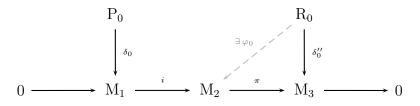
DIAGRAM FOR THE SNAKE LEMMA

<u>Lemma</u> ("Horseshoe Lemma") : Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be a short exact sequence of A-modules, and $P_{\bullet} \longrightarrow M_1$ and $R_{\bullet} \longrightarrow M_3$ projective resolutions of M_1 and M_3 respectively. Then there is a projective resolution $Q_{\bullet} \longrightarrow M_2$ with $Q_i = P_i \oplus R_i$ for all $i \ge 0$, such that the natural inclusion and quotient maps give a short exact sequence of chain complexes $P_{\bullet} \longrightarrow Q_{\bullet} \longrightarrow R_{\bullet}$.

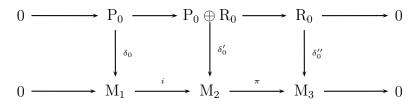
<u>Remark.</u> It is important in the application that Q_i splits as a direct sum — that is, we want more than just a projective resolution Q_{\bullet} fitting in an exact sequence.

Proof. By induction on *i*. We will do the cases i = 0, 1 (from which the general case is clear).

i = 0: We start with the following diagram, where δ_0 and δ_0'' are surjective



Since π is surjective, and R_0 projective, by the universal property of projective modules there exists a morphism $\varphi \colon R_0 \longrightarrow M_2$ such that $\delta''_0 = \pi \circ \varphi_0$. We define a morphism $\delta'_0 \colon P_0 \oplus R_0 \longrightarrow M_2$ by $\delta'_0 = (i \circ \delta_0) \oplus \varphi$, and thus obtain a commutative diagram with exact rows :

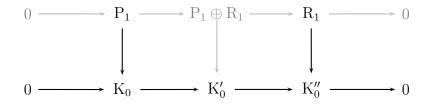


Let $K_0 = \text{Ker}(\delta_0)$, $K'_0 = \text{Ker}(\delta'_0)$, and $K''_0 = \text{Ker}(\delta''_0)$. Applying the snake lemma to the diagram above and using the surjectivity of δ_0 and δ''_0 we get the long exact sequence

 $0 \longrightarrow K_0 \longrightarrow K'_0 \longrightarrow K''_0 \longrightarrow 0 \longrightarrow \operatorname{Coker}(\delta'_0) \longrightarrow 0$

from which we conclude that δ'_0 is surjective, and that the kernels of δ_0 , δ'_0 , and δ''_0 form a short exact sequence.

i = 1: By hypothesis (i.e., the existence of the resolution) we have surjective maps $P_1 \longrightarrow K_0$ and $R_1 \longrightarrow K_0''$, i.e., we have a diagram



Repeating the construction from step i = 0, we obtain a surjective map $P_1 \oplus R_1 \longrightarrow K'_0$ which makes the diagram commute and the top row into a split exact sequence. Splicing the vertical maps with the inclusions $K_0 \hookrightarrow P_0$, $K'_0 \hookrightarrow P_0 \oplus R_0$, and $K''_0 \hookrightarrow R_0$, gives the next step of the resolution.

Passing to the kernels of the vertical maps (and again using the Snake Lemma to see that the kernels form an exact sequence) we continue inductively for $i \ge 2$.