1. In this problem we will compute some examples of Tor over the ring $\mathrm{A} = \mathbb{Z}$.

(a) Write down a projective resolution of $\mathbb{Z}/m\mathbb{Z}$, where m is a positive integer.

(b) For any abelian group G (which is therefore also a $\mathbb{Z}$-module) prove that

$$\mathrm{Tor}_1(\mathbb{Z}/m\mathbb{Z}, \mathrm{G}) = \left\{ g \in \mathrm{G} \;\middle|\; m \cdot g = 0 \right\}.$$

That is, that $\mathrm{Tor}_1(\mathbb{Z}/m\mathbb{Z}, \mathrm{G})$ calculates the m-torsion in G.

(c) Let m, n be positive integers. Prove the formula $\mathrm{Tor}_1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m, n)$. (We proved an identical formula for Tor_0 in **H4** Q2(a).)

(d) Let M be a finitely generated $\mathbb{Z}$-module. What does the structure theorem for finitely generated abelian groups say about M?

(e) Explain why M has a projective resolution consisting of only P_1 and P_0, with all other $\mathrm{P}_i = 0$.

(f) For finitely generated $\mathbb{Z}$-modules M and N, prove that $\mathrm{Tor}_i(\mathrm{M}, \mathrm{N}) = 0$ for all $i \geqslant 2$.

NOTE: The vanishing result in (f) actually holds for all $\mathbb{Z}$-modules; one can determine it from the result for finitely generated $\mathbb{Z}$-modules by an argument with a direct limit.

2. Let F be a <u>flat</u> A-module and $\mathrm{C}_\bullet$ any chain complex of A-modules. In this problem we will prove the useful formula $\mathrm{H}_i(\mathrm{C}_\bullet \otimes \mathrm{F}) = \mathrm{H}_i(\mathrm{C}_\bullet) \otimes \mathrm{F}$, valid for all i. Let us first recall some of the notation for a complex. Given a complex

$$\cdots \xrightarrow{\delta_{i+2}} \mathrm{C}_{i+1} \xrightarrow{\delta_{i+1}} \mathrm{C}_i \xrightarrow{\delta_i} \mathrm{C}_{i-1} \xrightarrow{\delta_{i-1}} \cdots$$

We have defined $\mathrm{Z}_i = \mathrm{Ker}(\delta_i) \subseteq \mathrm{C}_i$, $\mathrm{B}_i = \mathrm{Im}(\delta_{i+1}) \subseteq \mathrm{C}_i$, and $\mathrm{H}_i(\mathrm{C}_\bullet) = \mathrm{Z}_i/\mathrm{B}_i$. This gives us the following exact sequences

$$(\dagger) \qquad 0 \longrightarrow \mathrm{Z}_i \xrightarrow{j_i} \mathrm{C}_i \xrightarrow{\pi_i} \mathrm{B}_{i-1} \longrightarrow 0$$

and

$$(\ddagger) \qquad 0 \longrightarrow \mathrm{B}_i \xrightarrow{s_i} \mathrm{Z}_i \longrightarrow \mathrm{H}_i(\mathrm{C}_\bullet) \longrightarrow 0,$$

such that $\delta_i = j_{i-1} \circ s_{i-1} \circ \pi_i$. (Think about this formula to make sure it makes sense.)

(a) What sequences do we get if we tensor $(\dagger)$ and $(\ddagger)$ with F?

Use these two sequences, the fact that $\otimes \mathrm{F}$ is a functor, and the identity $\delta_i = j_{i-1} \circ s_{i-1} \circ \pi_i$ to prove that

(b) $\mathrm{Ker}(\delta_i \otimes \mathrm{Id_F}) = \mathrm{Z}_i \otimes \mathrm{F}$, and

(c) $\mathrm{Im}(\delta_{i+1} \otimes \mathrm{Id_F}) = \mathrm{B}_i \otimes \mathrm{F}$.

(d) Finally, show that $\mathrm{H}_i(\mathrm{C}_\bullet \otimes \mathrm{F}) = \mathrm{H}_i(\mathrm{C}_\bullet) \otimes \mathrm{F}$.

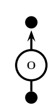
3. Now suppose that N is a module, not necessarily flat, and that $\mathrm{D}_\bullet$ is a complex of A-modules. What can we say about $\mathrm{H}_i(\mathrm{D}_\bullet \otimes \mathrm{N})$? and can it be computed from $\mathrm{H}_i(\mathrm{D}_\bullet)$ and N? In this problem we will see an answer under some extra assumptions (but an answer which is still useful in many applications). First, we will assume that the complex is a complex $\mathrm{P}_\bullet$ of projective modules. This does not mean that $\mathrm{P}_\bullet$ has to be a resolution : $\mathrm{P}_\bullet$ could have homology in every degree. Second we will assume that $\mathrm{P}_i = 0$ for all $i < 0$, so that $\mathrm{P}_\bullet$ is infinite in one direction only.

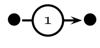
How do the homologies of $\mathrm{P}_\bullet$ and $\mathrm{P}_\bullet \otimes \mathrm{N}$ compare? The answer is that they are related by a spectral sequence. Specifically, there is a spectral sequence whose E_2 terms are $\mathrm{E}_2^{-p,-q} = \mathrm{Tor}_q(\mathrm{H}_p(\mathrm{P}_\bullet), \mathrm{N})$ and which computes $\mathrm{H}_{p+q}(\mathrm{P}_\bullet \otimes \mathrm{N})$. (The minus signs are there to orient the double complex in the directions we have been using in class.) In this problem we will derive this spectral sequence and one consequence.

Let $\mathrm{Q}_\bullet \longrightarrow \mathrm{N}$ be a projective resolution of N, and make a double complex by setting $\mathrm{C}^{-p,-q} = \mathrm{P}_p \otimes \mathrm{Q}_q$ for all $p, q \geqslant 0$.

(a) Draw a bit of this double complex. (Your drawing should be supported in the negative quadrant.) Draw enough pieces so that you get a feel for its shape, and what and where the terms are.

For each i, let $\mathrm{C}^i = \oplus_{p+q=i} \mathrm{C}^{p,q}$ be the i-th element of the total complex. We will now look at the two different spectral sequences computing the cohomology of $\mathrm{C}^\bullet$.

(b) Start with spectral sequence (II). Explain why after taking step  we have $\mathrm{P}_p \otimes \mathrm{N}$ in position $(-p, 0)$ and 0 in positions $(-p, -q)$ for $q > 0$. (Suggestion: Don't forget the formula from Q2; that each P_p is projective, and hence flat; and that $\mathrm{Q}_\bullet$ is a resolution.)

(c) After doing  , explain why we have 0 in positions $(-p, -q)$ for $q > 0$ and $\mathrm{H}_p(\mathrm{P}_\bullet \otimes \mathrm{N})$ in position $(-p, 0)$.

(d) Explain why the spectral sequence is stable from this point on, therefore why $\mathrm{H}^{-i}(\mathrm{C}^\bullet) = \mathrm{H}_i(\mathrm{P}_\bullet \otimes \mathrm{N})$ for all $i \geqslant 0$.

(e) Now we look at spectral sequence (I). Show that after taking step $\bullet - 0 \to \bullet$ that the term in position $(-p, -q)$ is $\mathrm{H}_p(\mathrm{P}_\bullet) \otimes \mathrm{Q}_q$. (SUGGESTION: The formula from Q2 is useful here too.)

(f) Explain why after taking step $\begin{smallmatrix}\bullet\\ \uparrow\\ 1\\ |\\ \bullet\end{smallmatrix}$, the term in position $(-p, -q)$ is $\mathrm{Tor}_q(\mathrm{H}_p(\mathrm{P}_\bullet), \mathrm{N})$.

(SUGGESTION: Use the definition of Tor and symmetry in the variables.)

This spectral sequence computes the cohomology of the total complex. By part (d) we know that the cohomology of the total complex in degree $-i$ is $\mathrm{H}^{-i}(\mathrm{C}^\bullet) = \mathrm{H}_i(\mathrm{P}_\bullet \otimes \mathrm{N})$. By part (f) we know there is a spectral sequence computing this whose E_2 term is $\mathrm{E}_2^{-p,-q} = \mathrm{Tor}_q(\mathrm{H}_p(\mathrm{P}_\bullet), \mathrm{N})$. That is, we have established the spectral sequence claimed above.

Finally, let us briefly consider one special case. Suppose that A is a ring such that $\mathrm{Tor}_i(-,-) = 0$ for all $i \geqslant 2$. (E.g., by Q1, $\mathbb{Z}$ is such a ring.)

(g) Explain why the spectral sequence is stable from E_2 onwards. (I.e, the step 2 maps and higher don't change the entries.)

This last result shows that in this case $\mathrm{H}_i(\mathrm{P}_\bullet \otimes \mathrm{N})$ is computed from $\mathrm{Tor}_0(\mathrm{H}_i(\mathrm{P}_\bullet), \mathrm{N}) = \mathrm{H}_i(\mathrm{P}_\bullet) \otimes \mathrm{N}$ and $\mathrm{Tor}_1(\mathrm{H}_{i-1}(\mathrm{P}_\bullet), \mathrm{N})$. This result (and Tor along with it) was first discovered in algebraic topology, and goes under the name of the *universal coefficient theorem* for homology.