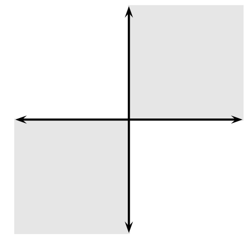
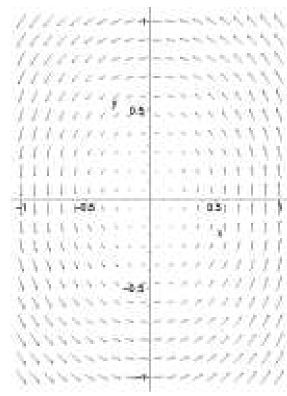
1. The domain is $U = \{ (x, y) \mid xy > 0 \}$, i.e., those pairs (x, y) where both x and y have the same sign, and neither are zero. This is sketched at right.

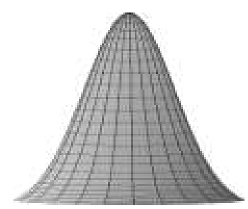


2.

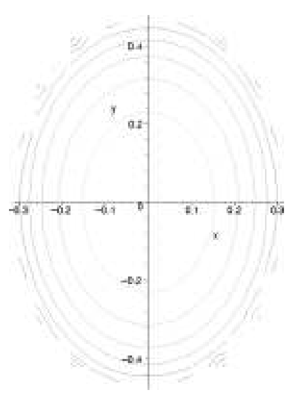
(a) At a point (x, y) the vector given by the vector field $\mathbf{F}(x, y) = (-y, x)$ points at right angles to the line connecting (x, y) to the origin, in the counterclockwise direction. The length of the vector is the same as the length of the line connecting (x, y) to the origin. If we were to look at a rotating disk from above, and at each point of the disk mark the instantaneous velocity vector, it would give this vector field.



(b) Since the function can be written as a function of $\sqrt{x^2 + y^2}$, it is symmetric under rotation. Restricting to the x-axis, we see that it is just the graph of e^{-x^2} rotated about the origin.



3. The function is only defined for $9x^2 + 4y^2 \leq 1$, and the function takes values in the interval $0 \leq c \leq 1$. For c in that range, the level curve $V(x, y) = c$ is given by the equation $c = \sqrt{1 - 9x^2 - 4y^2}$ or $9x^2 + 4y^2 = 1 - c^2$. These form a family of nested ellipses.



4.

(a) Since x^2+y^2+3 is continous, the limit is the same as plugging in $x=0$ and $y=0$, so the limit is 3.

(b) Since the numerator and denominator are continuous, and the denominator is nonzero at $(0,0)$, the limit is just the quotient $e^0/(0+1)=1$.

(c) Again, since both numerator and denominator are continuous, and the denominator is nonzero at $(0,0)$, the limit is again the quotient of the limits $0/2=0$.

(d) There are various ways to approach this question. Here is one option:

The Taylor series expansion of $\cos(x)$ is

$$\cos(x) = 1 - x^2/2 + x^4/4! - x^6/6! + x^8/8! - x^{10}/10! + \cdots ,$$

and so the expansion of $\cos(x) - 1 - x^2/2$ is

$$\cos(x) - 1 - x^2/2 = -x^2 + x^4/4! - x^6/6! + x^8/8! - x^{10}/10! + \cdots ,$$

Along the line $y=mx$, with points (x,mx), the function then looks like

$$\frac{-x^2 + x^4/4! - x^6/6! + x^8/8! - x^{10}/10! + \cdots}{(1+m^4)x^4}$$

which we can rewrite as

$$\frac{-1}{(1+m^4)x^2} + \frac{1}{4(1+m^4)} + \frac{-x^2/6! + x^4/8! - x^6/10! + \cdots}{(1+m^4)}$$

As x goes towards zero, the third term goes to zero, the middle term is constant, and the first term goes to $-\infty$. Therefore the limit doesn't exist. ($-\infty$ isn't allowed as a limit – it has to be a number. Or, if you'd like to allow $-\infty$, note that along the y-axis the function is zero, which would give another contradiction.)

(e) Along the line $y=mx$, with points (x,mx), the function is

$$\frac{(x-mx)^2}{x^2+m^2x^2} = \frac{(1-m)^2x^2}{(1+m^2)x^2} = \frac{(1-m)^2}{1+m^2},$$

i.e., the function is constant on lines $y=mx$, and hence if we go to $(0,0)$ along this line, the limit will also be $\frac{(1-m)^2}{1+m^2}$.

Since this number is different for different values of m, the limit doesn't exist.

5.

(a) The function $f(x, y)$ is not defined at $(0, 0)$, i.e., the domain is $\mathbb{R}^2 \setminus \{(0, 0)\}$. The limit $\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist. Like question 4(d) or 4(e), if we restrict to a line $y = mx$, we get the function

$$f(x, mx) = \frac{mx^2}{(1 + m^2)x^2} = \frac{m}{1 + m^2},$$

and so the limit as $x \to 0$ along this line is $\frac{m}{1+m^2}$. Since this depends on m, the limit doesn't exist.

(b) The function is not defined when $x + y = 0$, giving the domain as

$$\mathbb{R}^2 \setminus \{(x, y) \mid x + y = 0\}.$$

In order to understand what happens to the function as we approach this line, it's easier to note that this is secretly just a function of one variable, $x + y$. Using the coordinates $u = x + y$ and $v = x$ (or anything like that), we can write the function in terms of u and v as

$$g(u, v) = \frac{\sin(u)}{u}.$$

As (x, y) goes to $(0, 0)$, so does (u, v). In the formula, only the value of u matters, so we're really just worried about

$$\lim_{u\to 0} \frac{\sin(u)}{u}$$

since this is exactly the limit computing the derivative of $\sin(u)$ at $u = 0$, we see that the value is $\cos(0) = 1$.

Therefore, defining the function g to have the value 1 along the line $x + y = 1$ gives a continuous extension of the original function with domain all of $\mathbb{R}^2$. Since it is really a function of one variable, its graph looks like the one variable function, stretched in an extra direction.