1. We can parameterize the surface by

$$\begin{aligned} x(u,v) &= u & \mathbf{T}_u &= (1,0,2u) \\ y(u,v) &= v & \mathbf{T}_v &= (0,1,2v) \\ z(u,v) &= u^2+v^2 & \mathbf{T}_u \times \mathbf{T}_v = \mathbf{N} &= (-2u,-2v,1) \end{aligned}$$

with $0 \le u \le 3$, $0 \le v \le 2$.

The length of the normal vector (the "area scaling factor") is

$$\|\mathbf{N}\| = \sqrt{1+4u^2+4v^2}.$$

In terms of u and v, the function f becomes just $f = uv$, and so the surface integral is

$$\begin{aligned} \iint_S f \, dS &= \int_0^3 \int_0^2 uv\sqrt{1+4u^2+4v^2}\, dv\, du \\ &= \int_0^3 \frac{u}{12}\left(1+4u^2+4v^2\right)^{3/2}\Big|_{v=0}^{v=2} du \\ &= \frac{1}{12}\int_0^3 u(17+4u^2)^{3/2} - u(1+4u^2)^{3/2}\, du \\ &= \frac{1}{240}\left((17+4u^2)^{5/2} - (1+4u^2)^{5/2}\right)_{u=0}^{u=3} \\ &= \frac{1}{240}\left(53^{5/2} - 37^{5/2} - 17^{5/2} + 1\right) \approx 45.54978856\ldots \end{aligned}$$

2. We can use the usual parameterization for a sphere of radius r:

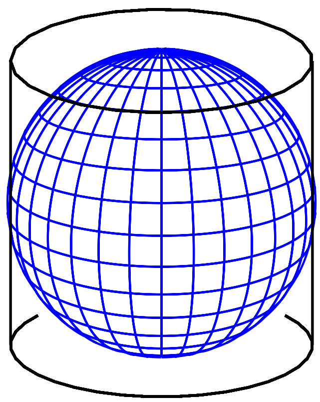
$$\begin{aligned} x(u,v) &= r\sin(\phi)\cos(\theta) & \mathbf{T}_\phi &= (r\cos(\phi)\cos(\theta),\ r\cos(\phi)\sin(\theta),\ -r\sin(\phi)) \\ y(u,v) &= r\sin(\phi)\sin(\theta) & \mathbf{T}_\theta &= (-r\sin(\phi)\sin(\theta),\ r\sin(\phi)\cos(\theta),\ 0) \\ z(u,v) &= r\cos(\phi) & \mathbf{N} &= (r^2\sin^2(\phi)\cos(\theta),\ r^2\sin^2(\phi)\sin(\theta),\ r^2\sin(\phi)\cos(\phi)) \end{aligned}$$

with $\phi_1 \le \phi \le \phi_2$, $0 \le \theta \le 2\pi$.

The length of the normal vector is $\|\mathbf{N}\| = r^2\sin(\phi)$. To find the area, we just integrate the function $f = 1$, so if S is the portion of the sphere between angles ϕ_1 and ϕ_2, we just need to compute

$$\iint_S 1\ dS = \int_0^{2\pi}\int_{\phi_1}^{\phi_2} r^2\sin(\phi)\,d\phi\,d\theta$$
$$= \int_0^{2\pi} r^2(\cos(\phi_2)-\cos(\phi_1))\,d\theta = 2\pi r^2(\cos(\phi_2)-\cos(\phi_1)).$$

There is a nice solution to this problem which has been known for over two thousand years. It's a wonderful theorem of Archimedes (287 B.C. – 212 B.C.) that if you surround a sphere with a cylinder of the same radius and height:



Then projection "sideways" from the vertical axis of the sphere onto the surface of the cylinder is an area preserving projection. I.e., whatever shape you draw on the surface of the sphere, when you project it onto the cylinder it may end up distorted, but it will still have the same area.

If we project the region of the problem onto the cylinder, it becomes a horizontal strip of the cylinder, of height $r(\cos(\phi_2)-\cos(\phi_1))$. Since the cylinder has radius r, this is of area $(2\pi r)(r(\cos(\phi_2)-\cos(\phi_1)) = 2\pi r^2(\cos(\phi_2)-\cos(\phi_1))$.

3. The parameterization of the helicoid is given:

$$\begin{aligned} x(u,v) &= u\cos(v) & \qquad \mathbf{T}_u &= (\cos(v),\, \sin(v),\, 0)\\ y(u,v) &= u\sin(v) & \mathbf{T}_v &= (-u\sin(v),\, u\cos(v),\, 1)\\ z(u,v) &= v & \mathbf{N} &= (\sin(v),\, -\cos(v),\, u)\end{aligned}$$

$$\text{with } 0 \le u \le 1,\ 0 \le v \le 4\pi.$$

Since $u \ge 0$ this means that the z-coordinate of the normal vector will always point upwards, and so this $\mathbf{N}$ is compatible with our chosen orientation.

In the (u, v) coordinates the vector field is $\mathbf{F} = (u\sin(v),\ -u\cos(v),\ uv\cos(v))$. The dot product of vector field and normal vector is $\mathbf{F}\cdot\mathbf{N} = u + u^2 v\cos(v)$.

If S is the oriented helicoid, this means that

$$\begin{aligned}
\iint_S \mathbf{F}\cdot dS &= \int_0^1\int_0^{4\pi} u + u^2 v\cos(v)\ dv\,du \\
&= \int_0^1 \left(uv + u^2 v\sin(v) + u^2\cos(v)\right)_{v=0}^{v=4\pi}\ du \\
&= \int_0^1 4\pi u\ du = 2\pi.
\end{aligned}$$

4. Let's use the usual parameterization of the top half of the unit sphere:

$$\begin{aligned}
x(u,v) &= \sin(\phi)\cos(\theta) & \mathbf{T}_\phi &= (\cos(\phi)\cos(\theta),\ \cos(\phi)\sin(\theta),\ -\sin(\phi)) \\
y(u,v) &= \sin(\phi)\sin(\theta) & \mathbf{T}_\theta &= (-\sin(\phi)\sin(\theta),\ \sin(\phi)\cos(\theta),\ 0) \\
z(u,v) &= \cos(\phi) & \mathbf{N} &= (\sin^2(\phi)\cos(\theta),\ \sin^2(\phi)\sin(\theta),\ \sin(\phi)\cos(\phi))
\end{aligned}$$

$$\text{with } 0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi.$$

In (ϕ,θ) coordinates the vector field is $\mathbf{F} = (\cos(\phi),\ \sin(\phi)\cos(\theta),\ \sin^2(\phi)\sin^2(\theta))$, with dot product $\mathbf{F}\cdot\mathbf{N} = \sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin(\theta)\cos(\theta) + \sin^3(\phi)\cos(\phi)\sin^2(\theta)$.

The integral looks like it's going to be a bit messy, but if we look at bit more carefully, we can see that we can omit two of the terms.

The range for the θ integral is 0 to 2π, and the integral of $\cos(\theta)$ over this range is zero. Similarly, the integral of $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$ is zero on this range. Therefore the first two terms contribute zero, and we only need to worry about the third term.

Using S for the top half of the sphere, oriented outwards, the flux integral is

$$\begin{aligned}
\iint_S \mathbf{F}\cdot dS &= \int_0^{2\pi}\int_0^{\pi/2} \sin^3(\phi)\cos(\phi)\sin^2(\theta)\,d\phi\,d\theta \\
&= \int_0^{2\pi} \frac{1}{4}\left(\sin^4(\phi)\sin^2(\theta)\right)_{\phi=0}^{\phi=\pi/2}\ d\theta \\
&= \frac{1}{4}\int_0^{2\pi} \sin^2(\theta)\,d\theta = \frac{\pi}{4}.
\end{aligned}$$

5. We can use the same parameterization from question 2:

$$\begin{array}{rclcrcl} x(u,v) &=& r\sin(\phi)\cos(\theta) & & \mathbf{T}_\phi &=& (r\cos(\phi)\cos(\theta),\ r\cos(\phi)\sin(\theta),\ -r\sin(\phi)) \\ y(u,v) &=& r\sin(\phi)\sin(\theta) & & \mathbf{T}_\theta &=& (-r\sin(\phi)\sin(\theta),\ r\sin(\phi)\cos(\theta),\ 0) \\ z(u,v) &=& r\cos(\phi) & & \mathbf{N} &=& (r^2\sin^2(\phi)\cos(\theta),\ r^2\sin^2(\phi)\sin(\theta),\ r^2\sin(\phi)\cos(\phi)) \end{array}$$

$$\text{with } 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

The normal vector always points outwards, so this $\mathbf{N}$ is compatible with our orientation. The easiest way to see this is to note that $\mathbf{N}$ is $r\sin(\phi)$ times the point $(x(\phi,\theta), y(\phi,\theta), z(\phi,\theta))$ on the sphere. Since in the range $0 \le \phi \le \pi$, $\sin(\phi)$ is always positive, $\mathbf{N}$ always points in the same direction as the (x, y, z) vector, i.e., outwards from the sphere.

In (ϕ, θ) coordinates, the vector field is $\mathbf{F} = (r^{-2}\sin(\phi)\cos(\theta),\ r^{-2}\sin(\phi)\sin(\theta),\ r^{-2}\cos(\phi))$. The dot product of the vector field and normal vector is

$$\mathbf{F}\cdot\mathbf{N} = \sin^3(\phi) + \sin(\phi)\cos^2(\phi) = \sin(\phi)(\sin^2(\phi) + \cos^2(\phi)) = \sin(\phi).$$

Therefore, with S the unit sphere oriented outwards, we have

$$\iint_S \mathbf{F}\cdot\, dS \;=\; \int_0^{2\pi}\int_0^{\pi} \sin(\phi)\, d\phi\, d\theta = 4\pi.$$

Writing $\mathbf{F} = (F_1,\ F_2,\ F_3)$, we have

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}, \\ \frac{\partial F_2}{\partial y} &= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}, \quad \text{and} \\ \frac{\partial F_3}{\partial z} &= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}, \end{aligned}$$

so that $\mathrm{Div}(\mathbf{F}) = \dfrac{3-3}{(x^2+y^2+z^2)^{3/2}} = 0.$

This doesn't contradict the divergence theorem. The vector field $\mathbf{F}$ is not defined at the origin, and in order to apply the divergence theorem $\mathbf{F}$ has to be defined (and C^1) over the entire volume.

The fact that the integral is independent of the radius of the sphere is connected to the fact that the divergence is zero. The vector field $\mathbf{F}$ of this question is an important one. We'll see in the next homework assignment that it "detects" whether or not a surface S contains the origin, and (later in class) that it is connected to electromagnetism.