1. We have $\mathbf{F}(x, y) = (e^{x^2},\ y\sin(\pi x),\ xy)$, and $\mathbf{G}(u, v, w) = (\cos(uv), w - u^2)$.

(a) The point q is the point $\mathbf{F}(1, 1) = (e,\ 0,\ 1)$ in $\mathbb{R}^3$.

(b) The composite function is $(\mathbf{G} \circ \mathbf{F})(x, y) = \left(\cos(e^{x^2} y \sin(\pi x)),\ xy - e^{2x^2}\right)$. with derivative matrix

$$\mathbf{D}(\mathbf{G}{\circ}\mathbf{F}) = \begin{bmatrix} -\sin(e^{x^2} y\sin(\pi x))ye^{x^2}(2x\sin(\pi x) + \pi\cos(\pi x)) & -\sin(e^{x^2} y\sin(\pi x))e^{x^2}\sin(\pi x) \\ y - 4xe^{2x^2} & x \end{bmatrix}$$

At the point $(1, 1)$ this has value $\mathbf{D}(\mathbf{G} \circ \mathbf{F})(1, 1) = \begin{bmatrix} 0 & 0 \\ 1 - 4e^2 & 1 \end{bmatrix}$.

(c) We have $\mathbf{DF} = \begin{bmatrix} 2xe^{x^2} & 0 \\ \pi y\cos(\pi x) & \sin(\pi x) \\ y & x \end{bmatrix}$, which at the point $(1, 1)$ has value

$$\mathbf{DF}(1, 1) = \begin{bmatrix} 2e & 0 \\ -\pi & 0 \\ 1 & 1 \end{bmatrix}.$$

Similarly, $\mathbf{DG} = \begin{bmatrix} -v\sin(uv) & -u\sin(uv) & 0 \\ -2u & 0 & 1 \end{bmatrix}$, which at the point q is

$$\mathbf{DG}(q) = \begin{bmatrix} 0 & 0 & 0 \\ -2e & 0 & 1 \end{bmatrix}.$$

(d) Multiplying, we get

$$\mathbf{DG}(q)\mathbf{DF}(1, 1) = \begin{bmatrix} 0 & 0 & 0 \\ -2e & 0 & 1 \end{bmatrix} \begin{bmatrix} 2e & 0 \\ -\pi & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 - 4e^2 & 1 \end{bmatrix},$$

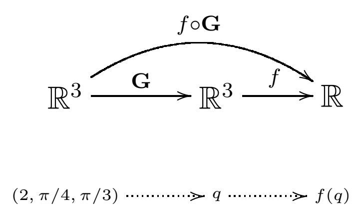
as the chain rule guarantees.

2. The way to deal with change of variables is to treat the process as a composition. We have a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, written in terms of (x, y, z) coordinates. We also have

a function $\mathbf{G}$ which translates from (ρ, α, θ) coordinates to (x, y, z) coordinates. This function is given by

$$\mathbf{G}(\rho, \alpha, \theta) = (\rho \sin(\alpha) \cos(\theta),\ \rho \sin(\alpha) \sin(\theta),\ \rho \cos(\alpha)) .$$

The function $\mathbf{G}$ is a function from $\mathbb{R}^3$ to $\mathbb{R}^3$, thinking of the first $\mathbb{R}^3$ as (ρ, α, θ) coordinates, and the second $\mathbb{R}^3$ as the (x, y, z) coordinates. When we want to understand f in terms of the (ρ, α, θ) coordinates, we're really talking about the composite function $f \circ \mathbf{G}$:



At the point q we're interested in, we know that $\mathbf{D}f(q) = \begin{bmatrix} \sqrt{3} & \sqrt{12} & -1 \end{bmatrix}$.

(a) The derivative matrix for $\mathbf{G}$ is

$$\mathbf{DG} = \begin{bmatrix} \sin(\alpha)\cos(\theta) & \rho\cos(\alpha)\cos(\theta) & -\rho\sin(\alpha)\sin(\theta) \\ \sin(\alpha)\sin(\theta) & \rho\cos(\alpha)\sin(\theta) & \rho\sin(\alpha)\cos(\theta) \\ \cos(\alpha) & -\rho\sin(\alpha) & 0 \end{bmatrix},$$

which at the point $(2, \pi/4, \pi/3)$ gives

$$\mathbf{DG}(2, \pi/4, \pi/3) = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\sqrt{2} & 0 \end{bmatrix}.$$

By the chain rule, $\mathbf{D}(f \circ \mathbf{G})(2, \pi/4, \pi/3) = \mathbf{D}f(q)\, \mathbf{DG}(2, \pi/3, \pi/4)$, or

$$\begin{aligned} \mathbf{D}(\mathbf{G} \circ f)(2, \pi/4, \pi/3) &= \begin{bmatrix} \sqrt{3} & \sqrt{12} & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\sqrt{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{6}}{4} + \sqrt{2}, & \frac{\sqrt{6}}{2} + 4\sqrt{2}, & -\frac{3}{\sqrt{2}} + \sqrt{6} \end{bmatrix} \end{aligned}$$

from which we can read off that (in this bad notation):

$$\begin{aligned}\frac{\partial f}{\partial \rho}(q) &= \frac{\sqrt{6}}{4}+\sqrt{2}, \\ \frac{\partial f}{\partial \alpha}(q) &= \frac{\sqrt{6}}{2}+4\sqrt{2}, \text{ and} \\ \frac{\partial f}{\partial \theta}(q) &= -\frac{3}{\sqrt{2}}+\sqrt{6}.\end{aligned}$$

(b) We want to move from $(2, \pi/3, \pi/4)$ in direction $\vec{v} = (v_\rho, v_\alpha, v_\theta)$ so that the instantaneous rate of change of f is zero. We also want v_ρ to be zero.

This means (using the expression for the derivative in the direction $\vec{v}$) that we want

$$\left(\frac{\sqrt{6}}{4}+\sqrt{2}\right)0+\left(\frac{\sqrt{6}}{2}+4\sqrt{2}\right)v_\alpha+\left(-\frac{3}{\sqrt{2}}+\sqrt{6}\right)v_\theta = 0.$$

The end result (after multiplying by $\sqrt{6}$ to make the equation slightly cleaner) is that we want to take $(0, v_\alpha, v_\theta)$ to be any multiple of the vector

$$\left(0,\ 3\sqrt{3}-6,\ 3+8\sqrt{3}\right).$$

(The numbers in this question weren't intended to be so awkward – I made a mistake in setting up the problem.)

3.

(a) $u_t = b\, e^{ax+bt}$ and $u_{xx} = a^2\, e^{ax+bt}$, so it's certainly true that $u_t = (b/a^2)\, u_{xx}$.

(b)

$$\begin{aligned}u_t &= -\tfrac{1}{2}t^{-3/2}e^{-x^2/t}+x^2t^{-5/2}e^{-x^2/t}, \text{ and} \\ u_x &= -2xt^{-3/2}e^{-x^2/t}, \text{ giving} \\ u_{xx} &= -2t^{-3/2}e^{-x^2/t}+4x^2t^{-5/2}e^{-x^2/t}\end{aligned}$$

and it's again clear that $u_t = \frac{1}{4}u_{xx}$.

(c) If f is a C^2 function, then $f_{xy} = f_{yx}$. If $f_x = e^x + xy$ then this gives $f_{xy} = \frac{d}{dy}f_x = x$. On the other hand, if $f_y = e^x + xy$, then $f_{yx} = \frac{d}{dx}f_y = e^x + y$. Since these aren't equal, there is no C^2 function with those x and y derivatives.

4. The equation $x^2 + y^2 = 1$ in $\mathbb{R}^3$ describes a cylinder of radius 1, with central axis the z-axis. Slicing with the plane $x + z = 1$ gives an ellipse contained in that plane.

In order to find a parameterization, we could first just parameterize the x and y values in the usual way:

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \end{aligned}$$

and then work out what the z coordinate should be. From the equation $x + z = 1$, or $z = 1 - x$ we see that

$$z(t) = 1 - \cos(t)$$

gives us a parameterization. (For, say $t \in [0, 2\pi]$).

5.

(a) Since $1/(x^2 + y^2)$ can be written solely in terms of $\sqrt{x^2 + y^2}$, the graph is rotationally symmetric. Restricting to the slice $y = 0$, we see that this is the graph of $z = 1/x^2$, rotated around the z axis.

(b) The parameterized curve satisfies $(x(t))^2 + (y(t))^2 = e^{2t}(\cos^2(t) + \sin^2(t)) = e^{2t}$, so $1/((x(t))^2 + (y(t))^2) = e^{-2t} = z(t)$, showing that the curve lies on the graph.

(c) Viewed from above the curve is a spiral with an exponentially increasing radius. From part (b) it lies on the graph. A piece of the curve is sketched on the graph above. (Note: some license was taken with this sketch. The radius increases so quickly that it's difficult to get an accurate sketch showing the features of the curve. In the picture above I increased the rate at which the curve rotates around the origin to be able to show how the curve spirals around on a reasonable piece of the graph.)