1. The operator Curl takes a vector field in $\mathbb{R}^3$ as input, and produces a vector field as output; Div takes a vector field as input and produces a function as output; grad takes a function as input and produces a vector field as output.

If f is a function, and $\mathbf{F}$ and $\mathbf{G}$ vector fields, then in terms of mathematical "grammar", the following expressions are

(a) grad(grad(f)): meaningless – grad(f) is a vector field, and so we can't take its gradient.

(b) $\mathrm{Curl}(\mathrm{grad}(f)) - \mathbf{F}$: a vector field. More precisely, it's the vector field $-\mathbf{F}$ if f is a C^2 function, since $\mathrm{Curl}(\mathrm{grad}(f)) = 0$.

(c) $\mathrm{Curl}(\mathrm{Curl}(\mathbf{F})) - \mathbf{G}$: a vector field. Since Curl($\mathbf{F}$) is a vector field, we can take its curl, and get another vector field, from which we can subtract $\mathbf{G}$.

(d) $\mathrm{Curl}(\mathbf{F}) \cdot \mathbf{G}$: a function. Since Curl($\mathbf{F}$) is a vector field, its dot product with the vector field $\mathbf{F}$ is a function.

(e) Div(Div($\mathbf{F}$)): meaningless. Div($\mathbf{F}$) is a function, and so we can't take its divergence.

(f) Div(Curl(grad(f))): A function, although not a very exciting one. If f is a C^2 function, then $\mathrm{Curl}(\mathrm{grad}(f)) = 0$. On the other hand for any C^2 vector field $\mathbf{F}$, $\mathrm{Div}(\mathrm{Curl}(\mathbf{F})) = 0$. In the diagram where "any two operators in a row are zero", this is the combination of all three of them in order. As long as f is a nice function, this is as zero as zero can be.

2.

(a) $\mathbf{F}(x, y, z) = (x,\ y,\ z)$, $\mathrm{Curl}(\mathbf{F}) = (0, 0, 0)$, $\mathrm{Div}(\mathbf{F}) = 3$,

(b) $\mathbf{F}(x, y, z) = (yz,\ xz,\ xy)$, $\mathrm{Curl}(\mathbf{F}) = (0, 0, 0)$, $\mathrm{Div}(\mathbf{F}) = 0$.

(c) $\mathbf{F}(x, y, z) = (3x^2y,\ x^3 + y^3,\ z^4)$, $\mathrm{Curl}(\mathbf{F}) = (0, 0, 0)$, $\mathrm{Div}(\mathbf{F}) = 6xy + 3y^2 + 4z^3$.

(d) $\mathbf{F}(x, y, z) = (e^x \cos(y) + z^2,\ e^x \sin(y) + xz,\ xy)$, $\mathrm{Curl}(\mathbf{F}) = (0, 2z - y, z + 2e^x \sin(y))$, $\mathrm{Div}(\mathbf{F}) = 2e^x \cos(x)$.

3.

(a) For $\mathbf{G} = (xy^2,\ yz^2,\ zx^2)$, if there were a vector field $\mathbf{F}$ with $\mathbf{G} = \mathrm{Curl}(\mathbf{F})$, then we would have $\mathrm{Div}(\mathbf{G}) = 0$. Since $\mathrm{Div}(\mathbf{G}) = y^2 + z^2 + x^2 \neq 0$ (as functions) there is no such vector field $\mathbf{F}$, i.e., $\mathbf{G}$ is not the curl of any vector field $\mathbf{F}$.

(b) For $\mathbf{G} = (2, 1, 3)$, we do have $\mathrm{Div}(\mathbf{G}) = 0$, and so there is some vector field $\mathbf{F}$ with $\mathrm{Curl}(\mathbf{F}) = \mathbf{G}$. Since all the entries of $\mathbf{G}$ are constant, and since computing Curl involves taking first derivatives, this suggests looking for a vector field $\mathbf{F}$ whose entries are linear functions of x, y, and z.

For a vector field $\mathbf{F}$ of the form $\mathbf{F}(x, y, z) = (ay + bz,\ cx + dz,\ ex + fy)$, we have $\mathrm{Curl}(\mathbf{F}) = (f - d,\ b - e,\ c - a)$. Picking any a, b, c, d, e, f with $f - d = 2$, $b - e = 1$, and $c - a = 3$ will give an $\mathbf{F}$ with $\mathrm{Curl}(\mathbf{F}) = \mathbf{G}$, for instance,

$$\mathbf{F}(x, y, z) = (z,\ 3x,\ 2y)$$

will work.

4.

(a) For a vector field of the form $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$, all the partial derivatives involved in computing the curl are zero, hence the curl is zero.

(b) For a vector field of the form $\mathbf{F}(x, y, z) = (f(y, z), g(x, z), h(x, y))$, all the partial derivatives involved in computing the divergence are zero, hence the divergence is zero.

(c) If $\mathbf{F}(x, y, z) = (3x - y + az,\ bx - z,\ 4x + cy)$, then

$$\mathrm{Curl}(\mathbf{F}) = [c + 1,\ a - 4,\ b + 1],$$

(for instance, we could just reuse the formula from the solution of 3(b), or compute directly). In order for this to be zero, we need $a = 4$, $b = -1$, and $c = -1$.

A function f with $\mathrm{grad}(f) = (3x - y + 4z,\ -x - z,\ 4x - y)$ is

$$f(x, y, z) = \tfrac{3}{2}x^2 - xy + 4xz - yz.$$

(d) If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ with $\mathrm{Curl}(\mathbf{F}) = 0$, then we know that there is a function f with $\mathrm{grad}(f) = \mathbf{F}$. By hypothesis, $\mathrm{Div}(\mathbf{F}) = 0$ too, so $\mathrm{Div}(\mathrm{grad}(f)) = 0$ since $\mathrm{grad}(f) = \mathbf{F}$. But $\mathrm{Div}(\mathrm{grad}(f)) = \Delta f$ by definition. Therefore $\Delta f = 0$, and so $\mathbf{F}$ is the gradient of a harmonic function.

5. It seems easiest to use the clue in question 4 parts (a) and (b) to try and decompose these vector fields, i.e., to write each $\mathbf{F}$ as a sum of two vector fields, one of the general form in 4(a) and one of the general form in 4(b).

(a) If $\mathbf{F}(x, y, z) = (x^2 + e^{yz},\ x^2 z^2,\ \sin(z))$, we can write this as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ with

$$\mathbf{F}_1(x, y, z) = (x^2,\ 0,\ \sin(z)), \text{ and } \mathbf{F}_2(x, y, z) = (e^{yz},\ x^2 z^2,\ 0).$$

By the calculation in question 4, $\mathbf{F}_1$ is irrotational and $\mathbf{F}_2$ is incompressible.

(b) There is a slight complication in this part, if $\mathbf{F}(x, y, z) = (x+y+z,\ y^2+1,\ \ln(xyz))$, then we want to find the same kind of decomposition as in part (a). It's tempting (and the right idea) to use the identity $\ln(xyz) = \ln(xy) + \ln(z)$, but this ignores a subtle point: using this identity reduces the domain of definition of the function.

The function $\ln(xyz)$ is defined as long as $xyz > 0$, i.e., in the eight octants in $\mathbb{R}^3$, the function $\ln(xyz)$ is defined in four of them. The common domain of $\ln(z)$ and $\ln(xy)$ is only two of the octants, and so we'd be losing half of the domain of our vector field.

The right identity to use is

$$\ln(xyz) = \begin{cases} \ln(xy) + \ln(z) & \text{if } z > 0 \\ \ln(-xy) + \ln(-z) & \text{if } z < 0 \end{cases}$$

which makes sense on all the domain of $\ln(xyz)$.

With that in mind, one possible decomposition is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ with

$$\mathbf{F}_1 = \begin{cases} (x,\ y^2 + 1,\ \ln(z)) & \text{if } z > 0 \\ (x,\ y^2 + 1,\ \ln(-z)) & \text{if } z < 0 \end{cases}$$

and

$$\mathbf{F}_2 = \begin{cases} (y + z,\ 0,\ \ln(xy)) & \text{if } z > 0 \\ (y + z,\ 0,\ \ln(-xy)) & \text{if } z < 0 \end{cases}.$$

The decomposition is not unique. If $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ is a decomposition with $\mathbf{F}_1$ irrotational, and $\mathbf{F}_2$ incompressible, and $\mathbf{G}$ is any vector field with $\mathrm{Curl}(\mathbf{G}) = 0$ and $\mathrm{Div}(\mathbf{G}) = 0$, then $\mathbf{F} = (\mathbf{F}_1 + \mathbf{G}) + (\mathbf{F}_2 - \mathbf{G})$ is another such decomposition.

We even know how to find such vector fields $\mathbf{G}$: by 4(d) these vector fields are exactly the vector fields of harmonic functions. There are therefore infinitely many possible decompositions into a sum of irrotational and incompressible vector fields.