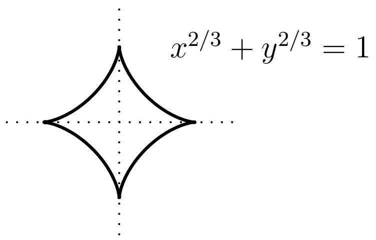
1. If f is a function on $\mathbb{R}^3$, and $\mathbf{F} = (F_1, F_2, F_3)$ a vector field on $\mathbb{R}^3$, then $f\mathbf{F} = (fF_1,\, fF_2,\, fF_3)$, so

$$\begin{aligned}
\mathrm{Div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3) \\
&= \left(\frac{\partial f}{\partial x}F_1 + f\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial f}{\partial y}F_2 + f\frac{\partial F_2}{\partial y}\right) + \left(\frac{\partial f}{\partial z}F_3 + f\frac{\partial F_3}{\partial z}\right) \\
&= \left(f\frac{\partial F_1}{\partial x} + f\frac{\partial F_2}{\partial y} + f\frac{\partial F_3}{\partial z}\right) + \left(\frac{\partial f}{\partial x}F_1 + \frac{\partial f}{\partial y}F_2 + \frac{\partial f}{\partial z}F_3\right) \\
&= f\,\mathrm{Div}(\mathbf{F}) + \mathbf{F}\cdot \mathrm{grad}(f)
\end{aligned}$$

2. One parameterization, which is just a variant of the way we parameterize the circle, is to define

$$\begin{aligned}
x(t) &= \cos^3(t) \\
y(t) &= \sin^3(t).
\end{aligned}$$



Then it's certainly true that $(x(t))^{2/3} + (y(t))^{2/3} = \cos^2(t) + \sin^2(t) = 1$, so this is a parameterization of the curve. Both $\cos^3(t)$ and $\sin^3(t)$ have derivatives of all orders, so this is a C^∞ parameterization. It is therefore also continuous and differentiable. (The curve might not look like it should have such a nice parameterization, but it does.)

3. We want to compute the integral of the function $f(x, y) = xy - x - y + 1 = (x-1)(y-1)$ along different curves connecting $(1, 0)$ and $(0, 1)$.

(a) $\mathbf{c}_1$, the circular arc $\mathbf{c}_1(t) = (\cos(t),\, \sin(t))$, $0 \le t \le \pi/2$.

The speed is $\|\mathbf{c}_1'(t)\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$, and $f(\mathbf{c}_1(t)) = \cos(t)\sin(t) - \cos(t) - \sin(t) + 1$, so the integral becomes

$$\begin{aligned}
\int_{\mathbf{c}_1} f\, ds &= \int_0^{\pi/2} \sin(t)\cos(t) - \cos(t) - \sin(t) + 1\, dt \\
&= \left.\frac{\sin^2(t)}{2} - \sin(t) + \cos(t) + t\right|_0^{\pi/2} \\
&= \left(\tfrac{1}{2} - 1 + 0 + \tfrac{\pi}{2}\right) - \left(\tfrac{0}{2} - 0 + 1 + 0\right) = \tfrac{\pi-3}{2}.
\end{aligned}$$

(b) $\mathbf{c}_2$, the straight line segment $\mathbf{c}_2(t) = (1-t, t)$, $0 \le t \le 1$.

The speed is $\|\mathbf{c}_2'(t)\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$, and $f(\mathbf{c}_2(t)) = (1-t)t - (1-t) - t + 1 = t - t^2$, so the integral becomes

$$\begin{aligned} \int_{\mathbf{c}_2} f\,ds &= \int_0^1 \sqrt{2}(t-t^2)\,dt \\ &= \sqrt{2}\left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Bigg|_{t=0}^{t=1} = \frac{\sqrt{2}}{6}. \end{aligned}$$

(c) $\mathbf{c}_3$, from $(1,0)$ horizontally to the origin, then vertically to $(0,1)$.

This curves is parameterized in two pieces. The first piece is $\mathbf{c}_{3,1}(t) = (1-t, 0)$ for $0 \le t \le 1$.

The speed is $\|\mathbf{c}_{3,1}'(t)\| = \sqrt{(-1)^2 + (0)^2} = 1$, and $f(\mathbf{c}_{3,1}(t)) = t$, so the first piece of the integral becomes

$$\int_{\mathbf{c}_{3,1}} f\,ds = \int_0^1 t\,dt = \frac{1}{2}.$$

The second piece is $\mathbf{c}_{3,2}(t) = (0, t)$ for $0 \le t \le 1$.

The speed is $\|\mathbf{c}_{3,2}'(t)\| = \sqrt{(0)^2 + (1)^2} = 1$, and $f(\mathbf{c}_{3,2}(t)) = 1 - t$, so the second piece of the integral becomes

$$\int_{\mathbf{c}_{3,2}} f\,ds = \int_0^1 (1-t)\,dt = \frac{1}{2}.$$

The total integral along $\mathbf{c}_3$ is just the sum of these two terms:

$$\int_{\mathbf{c}_3} f\,ds = \frac{1}{2} + \frac{1}{2} = 1.$$

(d) $\mathbf{c}_4$, from $(1,0)$ vertically to $(1,1)$, then horizontally to $(0,1)$.

This is again something we compute in two pieces. The first piece is $\mathbf{c}_{4,1}(t) = (1, t)$ for $0 \le t \le 1$.

The speed is $\|\mathbf{c}_{4,1}'(t)\| = \sqrt{(0)^2 + (1)^2} = 1$, and $f(\mathbf{c}_{4,1}(t)) = 0$, so the first piece of the integral is just

$$\int_{\mathbf{c}_{4,1}} f\,ds = \int_0^1 0\ dt = 0.$$

The second piece is $\mathbf{c}_{4,2}(t) = (1-t, 1)$ for $0 \le t \le 1$.

The speed is $\|\mathbf{c}'_{4,2}(t)\| = \sqrt{(-1)^2 + (0)^2} = 1$, and $f(\mathbf{c}_{4,2}(t)) = 0$ again, so again we have

$$\int_{\mathbf{c}_{4,2}} f\, ds = \int_0^1 0\, dt = 0.$$

The total integral along $\mathbf{c}_3$ is the sum

$$\int_{\mathbf{c}_4} f\, ds = 0 + 0 = 0.$$

(e) $\mathbf{c}_5$, the circular arc $\mathbf{c}_5(t) = (\cos(t), -\sin(t))$, $0 \le t \le 3\pi/2$.

The speed is $\|\mathbf{c}'_5(t)\| = \sqrt{(-\sin(t))^2 + (-\cos(t))^2} = 1$, and $f(\mathbf{c}_5(t)) = -\sin(t)\cos(t) - \cos(t) + \sin(t) + 1$, so the integral becomes

$$\begin{aligned}
\int_{\mathbf{c}} f\, ds &= \int_0^{3\pi/2} -\sin(t)\cos(t) - \cos(t) + \sin(t) + 1\, dt \\
&= \left.\frac{\cos^2(t)}{2} - \sin(t) - \cos(t) + t\right|_{t=0}^{t=3\pi/2} \\
&= \left(\tfrac{0}{2} - (-1) - 0 + \tfrac{3\pi}{2}\right) - \left(\tfrac{1}{2} - 0 - 1 + 0\right) = \tfrac{3\pi+3}{2}.
\end{aligned}$$

4. If $\mathbf{c}$ is the "twisted cubic:" $\mathbf{c}(t) = (t,\, t^2,\, t^3)$ for $t \in [0, 4]$, then $\mathbf{c}'(t) = (1,\, 2t,\, 3t^2)$, so $\|\mathbf{c}'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$. If the function f is $f(x, y, z) = \sqrt{9xz + 4y + 1}$ then

$$f(\mathbf{c}(t)) = f(t, t^2, t^3) = \sqrt{9 \cdot t \cdot t^3 + 4 \cdot t^2 + 1} = \sqrt{1 + 4t^2 + 9t^4}.$$

Therefore,

$$\begin{aligned}
\int_{\mathbf{c}} f\, ds &= \int_0^4 \left(\sqrt{1 + 4t^2 + 9t^4}\right)^2 dt \\
&= \int_0^4 1 + 4t^2 + 9t^4\, dt \\
&= \left. t + \tfrac{4}{3}t^3 + \tfrac{9}{5}t^5 \right|_{t=0}^{t=4} = \frac{28988}{15} = 1932 + \frac{8}{15}.
\end{aligned}$$

5. To find the average value of a function along a curve, we find its integral along the curve, and divide by the arclength. For the function $f(x, y, z) = xyz$, along the helix

$$\mathbf{c}(t) = (\sin(t),\, 8t,\, \cos(t)), \quad t \in [0, 6\pi],$$

the speed is $\|\mathbf{c}(t)\| = \sqrt{(\cos(t))^2 + (8)^2 + (-\sin(t))^2} = \sqrt{65}$, and $f(\mathbf{c}(t)) = 8t\cos(t)\sin(t)$. Therefore the integral of f along $\mathbf{c}$ is

$$\begin{aligned}
\int_{\mathbf{c}} f\,ds &= \int_0^{6\pi} 8\sqrt{65}\,t\cos(t)\sin(t)\,dt \\
&= \sqrt{65}\left(4t\sin^2(t) + \sin(2t) - 2t\right)\Bigg|_{t=0}^{t=6\pi} \\
&= \sqrt{65}\,(0+0-12\pi) - \sqrt{65}\,(0+0-0) = -12\sqrt{65}\pi.
\end{aligned}$$

The arclength is easier to find, since the speed is a constant $\sqrt{65}$, the arclength is

$$\int_0^{6\pi} \sqrt{65}\,dt = 6\sqrt{65}\,\pi.$$

The average value of f along $\mathbf{c}$ is therefore

$$\frac{-12\sqrt{65}\pi}{6\sqrt{65}\pi} = -2.$$