1. Connected or simply connected?

   (a) $\mathbb{R}^2$ with the circle $x^2 + y^2 = 1$ removed: neither connected nor simply connected.

   (b) $\mathbb{R}^3$ with the circle $x^2+y^2 = 1$, $z = 0$ removed: connected but not simply connected.

   (c) The set $\{(x, y) \mid 1 < x^2 + y^2 < 2\}$ in $\mathbb{R}^2$: connected but not simply connected.

   (d) $\mathbb{R}^3$ with the helix $(\cos(t), \sin(t), t)$, $t \in [0, \pi]$ removed: both connected and simply connected.

   (e) The set $\{(x, y) \mid x^2 - y^2 < 0\}$ in $\mathbb{R}^2$: simply connected but not connected.

2. For the three curves

   $\mathbf{c}_1$: The half-circle $(\cos(t), \sin(t), 0)$, $t \in [0, \pi]$, (*not* 2π!)
   $\mathbf{c}_2$: The segment $(-t, t^2 - 1, 1 - t^2)$ of a parabola, $t \in [-1, 1]$, and
   $\mathbf{c}_3$: The straight line $(-t, 0, 0)$, $t \in [-1, 1]$,

let's first calculate the velocity vectors, since we'll need them for both integrals.

$$\mathbf{c}_1'(t) = (-\sin(t), \cos(t), 0), \quad \mathbf{c}_2'(t) = (-1, 2t, -2t), \quad \mathbf{c}_3'(t) = (-1, 0, 0).$$

   (a) For $\mathbf{F} = (-y, x, z)$,

$$\begin{aligned}\int_{\mathbf{c}_1} \mathbf{F} \cdot ds &= \int_0^\pi \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt = \int_0^\pi (-\sin(t), \cos(t), 0) \cdot (-\sin(t), \cos(t), 0)\, dt \\ &= \int_0^\pi \sin^2(t) + \cos^2(t)\, dt = \pi.\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{c}_2} \mathbf{F} \cdot ds &= \int_{-1}^1 \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt = \int_{-1}^1 (1 - t^2, -t, 1 - t^2) \cdot (-1, 2t, -2t)\, dt \\ &= \int_{-1}^1 -1 - 2t - t^2 + 2t^3\, dt = \left(-t - t^2 - \frac{1}{3}t^3 + \frac{1}{2}t^4\right)_{t=-1}^{t=1} \\ &= -\frac{8}{3}\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{c}_3} \mathbf{F}\cdot ds &= \int_{-1}^{1} \mathbf{F}(\mathbf{c}_3(t))\cdot \mathbf{c}_3'(t)\,dt = \int_{-1}^{1}(0,\,-t,\,0)\cdot(-1,\,0,\,0)\,dt\\ &= \int_{-1}^{1} 0\ dt = 0.\end{aligned}$$

(b) For $\mathbf{G} = (e^{yz},\, xz\, e^{yz},\, xy\, e^{yz})$,

$$\begin{aligned}\int_{\mathbf{c}_1} \mathbf{G}\cdot ds &= \int_0^{\pi} \mathbf{G}(\mathbf{c}_1(t))\cdot \mathbf{c}_1'(t)\,dt = \int_0^{\pi}(e^0,\,0,\,\sin(t)\cos(t)\,e^0)\cdot(-\sin(t),\,\cos(t),\,0)\,dt\\ &= \int_0^{\pi} -\sin(t)\,dt = -2.\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{c}_2} \mathbf{G}\cdot ds &= \int_{-1}^{1} \mathbf{G}(\mathbf{c}_2(t))\cdot \mathbf{c}_2'(t)\,dt\\ &= \int_{-1}^{1}\left(e^{-(t^2-1)^2},\,(t^3-t)e^{-(t^2-1)^2},\,(t-t^3)e^{-(t^2-1)^2}\right)\cdot(-1,\,2t,\,-2t)\,dt\\ &= \int_{-1}^{1}(4t^4-4t^2-1)e^{-(t^2-1)^2}\,dt\\ &= -te^{-(t^2-1)^2}\Big|_{t=-1}^{t=1} = -1e^0-(-(-1)e^0) = -2.\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{c}_3} \mathbf{G}\cdot ds &= \int_{-1}^{1} \mathbf{G}(\mathbf{c}_3(t))\cdot \mathbf{c}_3'(t)\,dt = \int_{-1}^{1}(e^0,\,0,\,0)\cdot(-1,0,0)\,dt\\ &= \int_{-1}^{1} -1\,dt = -2.\end{aligned}$$

(c) Both $\mathbf{F}$ and $\mathbf{G}$ are defined on all of $\mathbb{R}^2$. For a conservative vector field (with any domain of definition), the path integrals connecting any two points p and q are independent of the path (which is assumed to lie in the domain of definition of the vector field).

The calculations in part (a) show that $\mathbf{F}$ cannot be a conservative vector field. The calculations in part (b) suggest that $\mathbf{G}$ might be a conservative vector field, and in fact it is. We can verify this by either

(i) computing that Curl($\mathbf{G}$) = (0, 0, 0), and using our theorems: since the domain of definition of $\mathbf{G}$ is simply connected, this is enough to imply that there must be a function g with $\mathbf{G} = \nabla g$. Or

(ii) finding the function g directly. In this case $g(x, y, z) = x\, e^{yz}$ is clearly a solution.

3. We're starting with the vector field $\mathbf{F}(x, y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$.

(a) The domain of definition of $\mathbf{F}$ is $\mathbb{R}^2$ minus the origin. It is connected but not simply connected.

(b) By the "$\mathbb{R}^2$ curl" of a vector field $\mathbf{F} = (F_1,\, F_2)$ we mean the function $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

For our vector field $\mathbf{F}$ we have

$$\begin{aligned} \frac{\partial F_2}{\partial x} &= \frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} \\ \frac{\partial F_1}{\partial y} &= \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \end{aligned}$$

So

$$\begin{aligned} \mathrm{Curl}(\mathbf{F}) &= \frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - \left(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) \\ &= \frac{-2}{x^2+y^2} + \frac{2x^2+2y^2}{(x^2+y^2)^2} \\ &= 0. \end{aligned}$$

(c) The curve $\mathbf{c}$ is the unit circle, oriented counterclockwise. We can use the usual parameterization $\mathbf{c}(t) = (\cos(t),\, \sin(t))$ for $t \in [0, 2\pi]$, with velocity vector $\mathbf{c}'(t) = (-\sin(t),\, \cos(t))$.

$$\begin{aligned} \int_{\mathbf{c}} \mathbf{F} \cdot ds &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\ &= \int_0^{2\pi} (\sin(t),\, -\cos(t)) \cdot (-\sin(t),\, \cos(t))\, dt \\ &= \int_0^{2\pi} -1\, dt = -2\pi \end{aligned}$$

(d) If $\mathbf{F} = \nabla f$ for some function f then for any closed curve $\mathbf{c}$ in the domain of $\mathbf{F}$ we would have to have

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds = 0.$$

One way to see this is to note that if we pick any point p of $\mathbf{c}$, we can consider $\mathbf{c}$ to be a curve that begins and ends at $\mathbf{c}$. Then, by the formula we proved in class (theorem 5.5 in the book)

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds = f(p) - f(p) = 0.$$

(e) From parts (c) and (d) we see that $\mathbf{F}$ cannot be the gradient of any function f.

The "curl test" is a local test (i.e., since it involves derivatives, it only involves very local information about the vector field $\mathbf{F}$) and only guarantees that locally there is a function f with $\mathbf{F} = \nabla f$. The problem of piecing these local possibilities of functions together to make a global function then depends on the topology of the domain of $\mathbf{F}$.

In this case since the domain of $\mathbf{F}$ is not simply connected, there is no guarantee that these local functions can be pieced together, and in fact our vector field $\mathbf{F}$ gives an example of a case where this patching is not possible.

4. Let f be the function $f(x, y) = x^2 y$, then

(a) $f(1, 1) - f(-1, -1) = 1^2 \cdot 1 - (-1)^2(-1) = 2.$

(b) $\mathbf{F} = \nabla f = (2xy,\, x^2).$

Now we look at the curves

$\mathbf{c}_1$: The half circle $(\sqrt{2}\cos(t),\, \sqrt{2}\sin(t))$, $t \in [-3\pi/4,\, \pi/4]$.
$\mathbf{c}_2$: The half circle $(\sqrt{2}\cos(t),\, -\sqrt{2}\sin(t))$, $t \in [3\pi/4,\, 7\pi/4]$.
$\mathbf{c}_3$: The straight line (t, t) $t \in [-1, 1]$.

with velocity vectors

$$\mathbf{c}_1'(t) = (-\sqrt{2}\sin(t),\, \sqrt{2}\cos(t)) \quad \mathbf{c}_2'(t) = (-\sqrt{2}\sin(t),\, -\sqrt{2}\cos(t)) \quad \mathbf{c}_3'(t) = (1, 1).$$

The integrals along these curves are

(c)

$$\begin{aligned}
\int_{\mathbf{c}_1} \mathbf{F} \cdot ds &= \int_{-3\pi/4}^{\pi/4} \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt \\
&= \int_{-3\pi/4}^{\pi/4} (4\sin(t)\cos(t),\ 2\cos^2(t)) \cdot (-\sqrt{2}\sin(t),\ \sqrt{2}\cos(t))\, dt \\
&= \int_{-3\pi/4}^{\pi/4} -4\sqrt{2}\sin^2(t)\cos(t) + 2\sqrt{2}\cos^3(t)\, dt \\
&= 2\sqrt{2}\cos^2(t)\sin(t)\Big|_{t=-3\pi/4}^{\pi/4} = 1 - (-1) = 2.
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbf{c}_2} \mathbf{F} \cdot ds &= \int_{3\pi/4}^{7\pi/4} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt \\
&= \int_{3\pi/4}^{7\pi/4} (-4\cos(t)\sin(t),\ 2\cos^2(t)) \cdot (-\sqrt{2}\sin(t),\ -\sqrt{2}\cos(t)) \\
&= \int_{3\pi/4}^{7\pi/4} 4\sqrt{2}\sin^2(t)\cos(t) - 2\sqrt{2}\cos^3(t)\, dt \\
&= -2\sqrt{2}\cos^2(t)\sin(t)\Big|_{t=3\pi/4}^{7\pi/4} = 1 - (-1) = 2.
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbf{c}_3} \mathbf{F} \cdot ds &= \int_{-1}^{1} \mathbf{F}(\mathbf{c}_3(t)) \cdot \mathbf{c}_3'(t)\, dt = \int_{-1}^{1} (2t^2,\ t^2) \cdot (1,1)\, dt \\
&= \int_{-1}^{1} 3t^2\, dt = t^3\Big|_{t=-1}^{t=1} = 1 - (-1)^3 = 2.
\end{aligned}$$

(d) The answers to (b) and (c) are of course all the same. The reason is the calculation we did in class (theorem 5.5 in the book again), if $\mathbf{F} = \nabla f$ is a conservative vector field, then for any oriented curve $\mathbf{c}$ in the domain of $\mathbf{F}$ starting at point q and ending at point p, we have

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds = f(p) - f(q).$$

The curves above all connect $q = (-1,-1)$ to $p = (1,1)$, which explains the calculations in part (c).