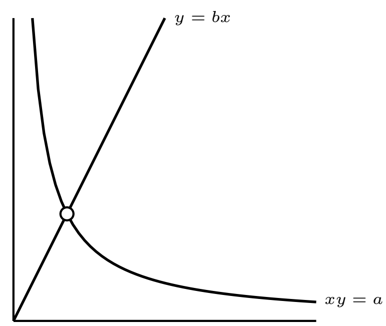
1.

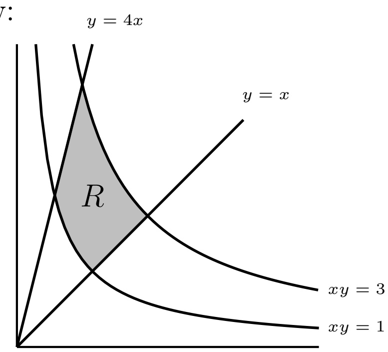
(a) Geometrically, the reason that $xy = a$ and $y = bx$ have only a single solution is that the line $y = bx$ intersects the curve $xy = a$ in only a single point in the positive quadrant:



Algebraically, we can see this by trying to solve for x and y: substituting $y = bx$ into $xy = a$ gives $x(bx) = a$ or $x = \sqrt{a/b}$ and $y = bx = \sqrt{ab}$ as the unique solutions with x and y positive.

(b) If $u = xy$ and $v = y/x$, the same steps as the algebraic solution in part (a) give $x = \sqrt{u/v}$ and $y = \sqrt{uv}$.

(c) The region is sketched below:



In terms of u, v coordinates, this region is a rectangle: $1 \le u \le 3$, $1 \le v \le 4$.

(c) The determinant of the derivative matrix for the change of variables is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v^{3/2}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}.$$

(d) The function $f(x,y) = x^3y^7$ is $(\sqrt{u/v})^3(\sqrt{uv})^7 = u^5v^2$ in terms of u and v.

In order to write the integral over the region R in terms of a u, v integral we have to:

(i) Work out the region R in terms of u, v coordinates.

(ii) Rewrite the function f in terms of u and v, and

(iii) Include the Jacobian factor to take the distortion in area due to the parameterization into account.

This gives us:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_1^3\int_1^4 u^5v^2\,\frac{1}{2v}\,dv\,du = \int_1^3 \left(\frac{1}{4}u^5v^2\right)_{v=1}^{v=4} du \\ &= \frac{15}{4}\int_1^3 u^5\,du = \frac{5}{8}\,u^6\Big|_{u=1}^{u=3} = 455.\end{aligned}$$

2. This time the region R is the one contained within the curves $xy = 1$, $xy = 2$, $x^2y = 1$, and $x^2y = 3$, and the function is $f(x,y) = x^2y^2$.

(a) If $u = x^2y$ and $v = xy$ then we can solve algebraically for x and y. Substituting, we get $u = x^2y = x(xy) = xv$ or $x = u/v$, which then gives $y = v^2/u$.

(b) In terms of u and v, the region R again becomes a rectangle $1 \le u \le 3$, $1 \le v \le 2$.

(c) $f = x^2y^2 = (u/v)^2(v^2/u)^2 = v^2$ (or, $f = (xy)^2 = v^2$, which is faster).

(d) The Jacobian factor is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{1}{v} & -\frac{u}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u}\end{matrix}\right| = \frac{1}{u}.$$

(e) Using the change of variables theorem, the integral becomes

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_1^3\int_1^2 v^2\frac{1}{u}\,dv\,du = \int_1^3 \left.\frac{v^3}{3u}\right|_{v=1}^{v=2} du \\ &= \frac{7}{3}\int_1^3 \frac{1}{u}\,du = \frac{7}{3}\ln(3).\end{aligned}$$

3.

(a) The region of integration is the region below the paraboloid $z = 8 - x^2 - y^2$ and above $z = -3$, restricted to the cylinder $x^2 + y^2 \leq 8$.

$$\int_{-\sqrt{8}}^{\sqrt{8}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{-3}^{8-x^2-y^2} 2\, dz\, dy\, dx = 2 \int_{-\sqrt{8}}^{\sqrt{8}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} (11 - x^2 - y^2)\, dy\, dx$$

$$= 2 \int_{-\sqrt{8}}^{\sqrt{8}} \left(11y - x^2 y - \frac{y^3}{3} \right)_{y=-\sqrt{8-x^2}}^{y=\sqrt{8-x^2}} dx = \frac{4}{3} \int_{x=-\sqrt{8}}^{x=\sqrt{8}} (25 - 2x^2)\sqrt{8 - x^2}\, dx$$

$$= \left(14x(8 - x^2)^{1/2} + \frac{2}{3} x(8 - x^2)^{3/2} + 112 \arcsin\left(\frac{x}{\sqrt{8}} \right) \right)_{x=-\sqrt{8}}^{x=\sqrt{8}} = 112\pi.$$

(b) The region of integration is the part of the unit ball in the positive octant.

The first step is easy:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} (2x - y)\, dz\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} (2x - y)\sqrt{1 - x^2 - y^2}\, dx\, dy$$

But this integral is somewhat awkward to work out. It might be better to split it into two parts:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} 2x\sqrt{1 - x^2 - y^2}\, dx\, dy = \frac{2}{3} \int_0^1 -(1 - x^2 - y^2)^{3/2} \Big|_{x=0}^{x=\sqrt{1-y^2}} dy$$

$$= \frac{2}{3} \int_0^1 (1 - y^2)^{3/2}\, dy = \frac{1}{12} \left((5y - 2y^3)\sqrt{1 - y^2} + 3 \arcsin(y) \right)_{y=0}^{y=1} = \frac{\pi}{8}.$$

To deal with the second half, we can switch the order of integration:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} -y\sqrt{1 - x^2 - y^2}\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-x^2}} -y\sqrt{1 - x^2 - y^2}\, dy\, dx$$

Which we recognize as the same integral (with the roles of x and y reversed) as we did in the first part, up to a factor of $-\frac{1}{2}$.

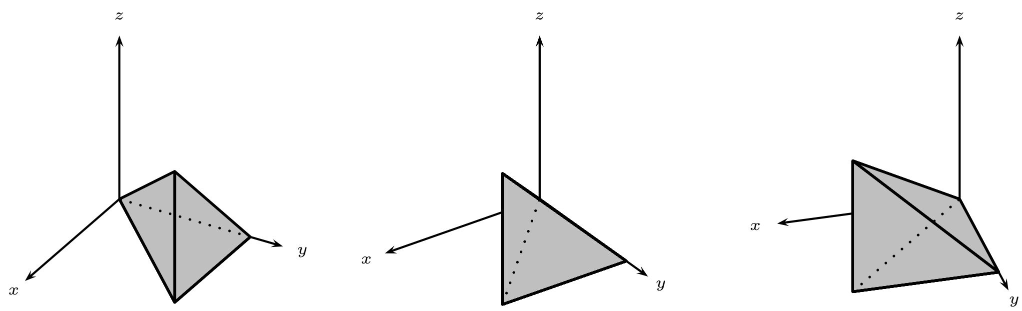
This means that we must have

$$\int_0^1 \int_0^{\sqrt{1-x^2}} -y\sqrt{1-x^2-y^2}\,dy\,dx = -\frac{1}{2}\left(\frac{\pi}{8}\right) = -\frac{\pi}{16}.$$

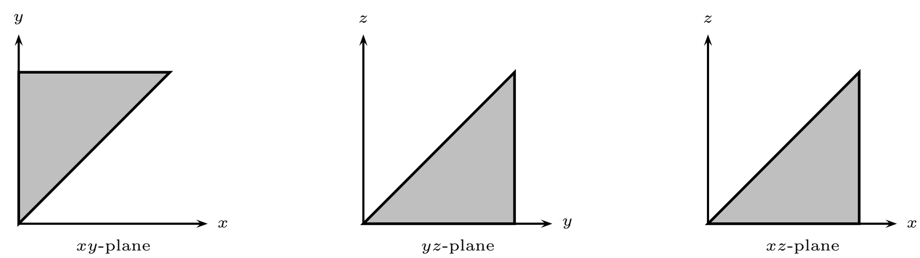
So that

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (2x-y)\sqrt{1-x^2-y^2}\,dx\,dy = \frac{\pi}{8} - \frac{\pi}{16} = \frac{\pi}{16}.$$

4. The region V of integration is a tetrahedron with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, and $(1,1,1)$. Here are three views of the region:



The shadows of V on the xy, yz and xz planes are shown below:



The six possible orders of integration are then

(a) $\int_0^1 \int_0^y \int_0^x x^2yz\,dz\,dx\,dy$ (b) $\int_0^1 \int_x^1 \int_0^x x^2yz\,dz\,dy\,dx$ (c) $\int_0^1 \int_0^y \int_z^y x^2yz\,dx\,dz\,dy$

(d) $\int_0^1 \int_z^1 \int_z^y x^2yz\,dx\,dy\,dz$ (e) $\int_0^1 \int_0^x \int_x^1 x^2yz\,dy\,dz\,dx$ (f) $\int_0^1 \int_z^1 \int_x^1 x^2yz\,dy\,dx\,dz$

5.

(a) The function is positive over the circle $x^2 + y^2 \leq 9$.

One possible parameterization is to parameterize the circle using polar coordinates, and then use the equation of the graph to get the z coordinate. This parameterization is

$$\begin{aligned} x(r,\theta) &= r\cos(\theta) \\ y(r,\theta) &= r\sin(\theta) \\ z(r,\theta) &= 9 - r^2 \end{aligned} \quad \text{with} \quad \begin{aligned} \mathbf{T}_r &= (\cos(\theta), \sin(\theta), -2r) \\ \mathbf{T}_\theta &= (-r\sin(\theta), r\cos(\theta), 0) \\ \mathbf{N} &= \mathbf{T}_r \times \mathbf{T}_\theta = (2r^2\cos(\theta), 2r^2\sin(\theta), r) \end{aligned}$$

where $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$.

We can also use the general form for the graph of a function:

$$\begin{aligned} x(u,v) &= u \\ y(u,v) &= v \\ z(u,v) &= f(u,v) \end{aligned} \quad \text{with} \quad \begin{aligned} \mathbf{T}_u &= (1, 0, f_u) \\ \mathbf{T}_v &= (0, 1, f_v) \\ \mathbf{N} &= \mathbf{T}_u \times \mathbf{T}_v = (-f_u, -f_v, 1) \end{aligned}$$

For $f(x,y) = 9 - x^2 - y^2$ this gives the normal vector $\mathbf{N} = (2u, 2v, 1)$, with (u,v) in the circle $u^2 + v^2 \leq 9$.

(b) The parameterizations of the piece of the paraboloid in the first octant are similiar.

Using polar coordinates:

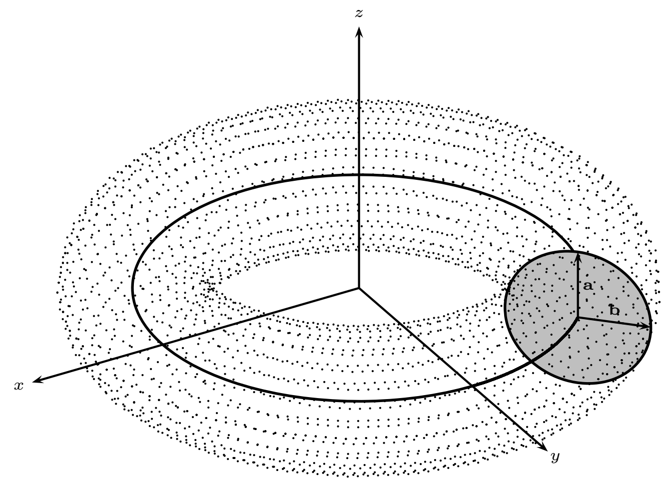
$$\begin{aligned} x(r,\theta) &= r\cos(\theta) \\ y(r,\theta) &= r\sin(\theta) \\ z(r,\theta) &= r^2 \end{aligned} \quad \text{with} \quad \begin{aligned} \mathbf{T}_r &= (\cos(\theta), \sin(\theta), 2r) \\ \mathbf{T}_\theta &= (-r\sin(\theta), r\cos(\theta), 0) \\ \mathbf{N} &= \mathbf{T}_r \times \mathbf{T}_\theta = (-2r^2\cos(\theta), -2r^2\sin(\theta), r) \end{aligned}$$

with $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi/2$.

Or, using the general form for the graph of a function above, we could use $x(u,v) = u$, $y(u,v) = v$, and $z(u,v) = f(u,v) = u^2 + v^2$. By the formulas from part (a), this gives

$$\mathbf{T}_u = (1, 0, 2u), \;\; \mathbf{T}_v = (0, 1, 2v), \;\text{ and } \mathbf{N} = (-2u, -2v, 1).$$

(c) This one is a little trickier to parameterize. The surface is a torus (i.e., a doughnut).



We can parameterize the center circle of the torus by $c(\theta) = (3\cos(\theta), 3\sin(\theta), 0)$.

In the slice of the torus around that point, we can draw two vectors which generate the circle (a "moving frame" around the center point c), $a(\theta) = (0, 0, 1)$ and $b(\theta) = (\cos(\theta), \sin(\theta), 0)$.

Now for any angle α, the linear combination $a(\theta)\cos(\alpha) + b(\theta)\sin(\alpha)$, when added to the center point $c(\theta)$ will give us a point on the circle around the center point $c(\theta)$. Putting this together, we can parameterize the torus by

$$\begin{aligned} x(\theta, \alpha) &= 3\cos(\theta) + \cos(\theta)\cos(\alpha) = (3 + \cos(\alpha))\cos(\theta) \\ y(\theta, \alpha) &= 3\sin(\theta) + \sin(\theta)\cos(\alpha) = (3 + \cos(\alpha))\sin(\theta) \\ z(\theta, \alpha) &= \sin(\alpha) \end{aligned}$$

Giving

$$\begin{aligned} \mathbf{T}_\theta &= \Big(-(3 + \cos(\alpha))\sin(\theta),\ (3 + \cos(\alpha))\cos(\theta),\ 0\Big) \\ \mathbf{T}_\alpha &= \Big(-\sin(\alpha)\cos(\theta),\ -\sin(\alpha)\sin(\theta),\ \cos(\alpha)\Big) \end{aligned}$$

and

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_\theta \times \mathbf{T}_\alpha \\ &= \Big((3 + \cos(\alpha))\cos(\alpha)\cos(\theta), (3 + \cos(\alpha))\cos(\alpha)\sin(\theta), (3 + \cos(\alpha))\sin(\alpha)\Big). \end{aligned}$$