Curves

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Curves are the images of (at least piecewise) continuous $\mathbb{R} \to \mathbb{R}^n$ functions (the functions themselves are referred to as *paths*). Different paths can produce the same curve: e.g. $\mathbf{x}_1(t) = (t, t^2)$ and $\mathbf{x}_2(t) = (t^3, t^6)$. Most of the time, we will analyize differentiable curves. Coordinatewise differentiable functions imply differentiable curves, but *the converse is not true*.

Excercise 0: Find a pair of non-differentiable functions $f(t)$and $g(t)$ so that $\mathbf{x}(t) = (f(t), g(t))$ yields a differentiable curve.

Often, the physical analogy for the path is *location* (in space) as a function of *time*. Hence, the derivative of the path $\mathbf{v}(t) = \frac{\partial \mathbf{x}}{\partial t}$ is often referred as velocity. The tangent of some curve C at some point $\mathbf{x}$ is a straight line (a curve itself!) L also containing $\mathbf{x}$, so that for $\mathbf{y} \in C$ and $\mathbf{z} \in L$:

$$\lim_{|\mathbf{y}-\mathbf{x}|\to 0} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} = \lim_{|\mathbf{z}-\mathbf{x}|\to 0} \frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|}$$

If C is defined as a function of t so that is coordinatewise differentiable and $\mathbf{x} = C(t_0)$ for some t_0 and $\mathbf{v}(t_0) \neq \underline{0}$, then $L(t) = \mathbf{x} + t\mathbf{v}(t_0)$ defines a tangent.

Excercise 1: Prove the above statement.

Problem 1: What if that's not the case? Find the tangent of the curve induced by $\mathbf{x}(t) = (t^5, t^3)$ in the origin.

Often, however, curves are *not* defined as functions. Finding such a description is called *parametrization* of the curve.

Problem 2: A curve C is defined as follows:

$$\mathbf{x} = (x, y) \in C : (x, y) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

Parametrize this curve. That is find $\mathbf{x}(t) = (x(t), y(t))$ such that
$\forall t \in \mathbb{R} : \mathbf{x}(t) \in C$ and $\forall(\mathbf{x} \in C)\exists t : \mathbf{x}(t) \in C$.

Problem 3: Find a tangent of C going through the point $\mathbf{y} = (3, 3)$.