Change of Coordinates and the Jacobian

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1. IT IS NOT THE CASE THAT $dx\,dy = dr\,d\theta$. IT IS THE CASE THAT $dx\,dy = rdr\,d\theta$. ONE DOES NOT SIMPLY REPLACE INFINITESIMALS.

2. We change variables in integration to: i) simplify the region of the integration ii) simplify the integrand, or iii) both.

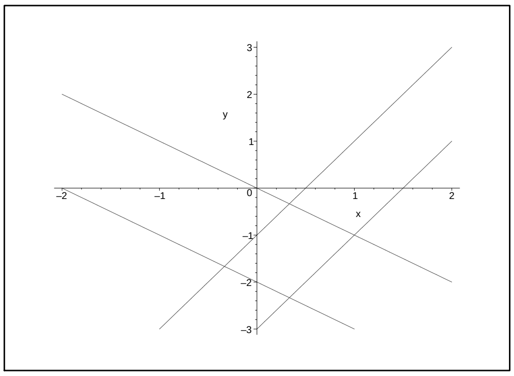
   (a) Simplify the region.

      i. What is the area of an ellipse?

$$
\begin{aligned}
A &= \int\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} dx\,dy \\
&= \int\int_{u^2+v^2=1} \left\| \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right\| du\,dv \\
&= ab\int\int_{u^2+v^2=1} du\,dv \\
&= ab\int_{r=0}^{1}\int_{\theta=0}^{2\pi} \left\| \begin{array}{cc} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{array} \right\| dr\,d\theta \\
&= ab\int_{r=0}^{1}\int_{\theta=0}^{2\pi} rdr\,d\theta \\
&= \frac{ab}{2}\int_{0}^{2\pi} d\theta \\
&= \pi ab
\end{aligned}
$$

      ii. Evaluate $\int\int_P 2x^2+xy-y^2\,dxdy$ where $P = \{(x,y)|1 \leq 2x-y \leq 3, -2 \leq x+y \leq 0\}$. Use $u = 2x-y$ and $v = x+y$. $P = T(R)$ where R is the rectangle $[1,3]\times[-2,0]$ in the uv plane. Observe that $2x^2+xy-y^2 = (2x+y)(x+y) = uv$ and $\frac{\partial(x,y)}{\partial(u,v)} = \left\| \begin{array}{cc} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{array} \right\| = \frac{1}{3}$

$$
\int\int_P 2x^2+xy-y^2\,dxdy = \frac{1}{3}\int_{u=1}^{3}\int_{v=-2}^{0} uv\,du\,dv
$$



$$
\begin{aligned}
&= \frac{1}{3}\int_{v=-2}^{0} \frac{1}{2}(9-1)v dv \\
&= -\frac{8}{3}
\end{aligned}
$$

(b) Simplify the integrand. Evaluate the integral $\int\int_R e^{\frac{x+y}{x-y}} dx\, dy$ where P is the trapazoid with vertices $(1,0),(2,0),(0,-2),(0,-1)$.
Use the transformation $u = x+y$ and $v = x-y$. So $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$. How does the trapazoid transform? From the vertices of the trapazoid we get the four bounding lines: $y = 0, x = 0, x-y = 2, x-y = 1$. We apply our transformation for x and y in terms of u and v to get the transformed lines: $u = v, u = -v, v = 2, v = 1$. The region becomes $T(P) = \{(u,v)|1 \le v \le 2, -v \le u \le v\}$. $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$. So:

$$
\begin{aligned}
\int\int_R e^{\frac{x+y}{x-y}} dx\, dy &= \frac{1}{2}\int_{v=1}^{2}\int_{u=-v}^{v} e^{\frac{u}{v}} du\, dv \\
&= \frac{1}{2}\int_{1}^{2}(e-e^{-1})v\, dv \\
&= \frac{3}{4}(e-e^{-1})
\end{aligned}
$$

Remember in changing from the xy plane to the uv plane **all** x's and y's must disappear. Change:

(a) integrand
(b) infinitesimals (remember Jacobian!)
(c) limits of integration

3. A very quick note on geometry. Feel free to ignore. Where does the Jacobian, a determinant, come from? Remember that the magnitude of a cross product is a measure of the area of the parallelogram spanned

by two vectors. The cross product can also be evaluated in terms of a formula involving the determinant. The two vectors we use are $\mathbf{T}_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0)$ and $\mathbf{T}_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0)$. These are the tangent vectors of the coordinate lines of x and y in terms of u and v. All this is related to area because an infinitesimal rectangle in the uv plane becomes an infinitesimal parallelogram in the xy plane where the areas are related by:

$$\|\Delta u \mathbf{T}_u \times \Delta v \mathbf{T}_v\| = \|\mathbf{T}_u \times \mathbf{T}_v\| \Delta u \Delta v$$