DUE DATE: DEC. 1, 2004

1. Find the area contained in the hypocycloid curve: $x^{2/3} + y^{2/3} = 1$.

2. Let V be the region in $\mathbb{R}^3$ described by $1 \leq x^2 + y^2 \leq 9$, $0 \leq z \leq 2$, and S its boundary surface, oriented outwards (outwards from V that is). Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (2x, xy^2, xyz).$$

(a) Sketch V. How many pieces does the boundary surface S have?

(b) Work out the flux integral $\iint_S \mathbf{F} \cdot dS$ directly by parameterizing each of the pieces of S and computing the integrals of each of them.

(c) Compute $\mathrm{Div}(\mathbf{F})$ and compute (by working it out) the integral of $\mathrm{Div}(\mathbf{F})$ over V. This answer should be the same as (b), by the Divergence theorem.

3. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = (y, z, x^2)$ on $\mathbb{R}^3$.

(a) Let S_1 be the top half of the unit sphere, oriented upwards. Compute $\iint_{S_1} \mathbf{F} \cdot dS$.

(b) Let S_2 be the disk $x^2 + y^2 \leq 1$ in the xy-plane, oriented upwards. Compute $\iint_{S_1} \mathbf{F} \cdot dS$.

(c) Find a vector field $\mathbf{G}$ with $\mathrm{Curl}(\mathbf{G}) = \mathbf{F}$.

(d) Let $\mathbf{c}$ be the unit circle $x^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise. Notice that $\mathbf{c}$ is the oriented boundary curve of both S_1 and S_2. Compute $\int_{\mathbf{c}} \mathbf{G} \cdot ds$.

(e) In general, if S_1 and S_2 are two oriented surfaces in $\mathbb{R}^3$, which have the same oriented boundary curve $\mathbf{c}$, and if $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ with $\mathrm{Div}(\mathbf{F}) = 0$, explain why $\iint_{S_1} \mathbf{F} \cdot dS = \iint_{S_2} \mathbf{F} \cdot dS$.

There are at least two possible explanations. One is to use Stokes' theorem and the ideas above, and another to use the divergence theorem. Don't forget to explain how the assumption that $\mathrm{Div}(\mathbf{F}) = 0$ enters into either argument.

4. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

(a) If $\mathbf{c}_1$ is the unit circle $x^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise, compute $\int_{\mathbf{c}_1} \mathbf{F} \cdot ds$.

(b) If $\mathbf{c}_2$ is the unit circle $x^2 + y^2 = 1$ in the plane $z = 3$, oriented counterclockwise, compute $\int_{\mathbf{c}_2} \mathbf{F} \cdot ds$.

(c) If $\mathbf{c}_3$ is the circle $(x-3)^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise, compute $\int_{\mathbf{c}_3} \mathbf{F} \cdot ds$.

(d) Compute that $\mathrm{Curl}(\mathbf{F}) = (0,0,0)$. Explain why Stokes' theorem predicts that the the integral in (c) is zero, and that the integrals in (a) and (b) are the same.

(HINT: To show that (c) should be zero, consider the disk $(x-3)^2+y^2 \leq 1$, $z = 0$. To show that (a) and (b) should be the same, consider the surface $x^2 + y^2 = 1$, $0 \leq z \leq 3$. Pay attention to the orientation of the surface and the boundary.)

(e) Why can't we use Stokes' theorem to conclude that that the integrals in (a) and (b) are also zero?

5. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right).$$

You can assume (or compute) that $\mathrm{Div}(\mathbf{F}) = 0$.

Similar to question 4, explain:

(a) If S is any closed surface which does not contain the origin, then $\iint_S \mathbf{F} \cdot dS = 0$.

(b) If S is any closed surface which does contain the origin, then $\iint_S \mathbf{F} \cdot dS = 4\pi$.

For part (b), it might be easier to first show that no matter what S is, the flux integral through S is the same as the flux integral through a sphere S_ϵ of radius ϵ contained in S (for ϵ small enough so that S_ϵ *is* contained in S). Then for the flux integral through S_ϵ, you can compute this directly, or use question 5 of the previous assignment.

Questions 4 and 5 are examples of the fact that integrals along special kinds of vector fields can detect topological (i.e., "shape") information. In question 4, integrating $\mathbf{F}$ around any closed curve $\mathbf{c}$ detects how many times $\mathbf{c}$ winds around the z-axis. In question 5, integrating $\mathbf{F}$ through any closed surface S detects whether or not the surface contains the origin.