Math 280 Homework Assignment 2

DUE DATE: SEPT. 29, 2004

1. Sketch the region $U = \{(x, y) \mid |y| \leq \sin(x)\}$, and describe the interior points and the boundary points.

2. Let $u(x, y, t) = e^{-2t} \sin(3x) \cos(2y)$ denote the vertical displacement of a vibrating membrane from the point (x, y) in the xy-plane at time t. Compute $u_x(x, y, t)$, $u_y(x, y, t)$, and $u_t(x, y, t)$ and give physical interpretations of these results.

3. If $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ is the function

$$\mathbf{F}(x, y) = \left(\sin(\pi x)\cos(\pi y),\ ye^{xy},\ x^2 + y^3\right),$$

compute the derivative matrix $\mathbf{DF}$ at $(1, 2)$. If (at $(1, 2)$) we go in the direction $\vec{v} = (3, -2)$, what are the instantaneous rates of change of the functions $\sin(\pi x)\cos(\pi y)$, ye^{xy}, and $x^2 + y^3$?

4. I'd like to know if the function

$$f(x, y) = \begin{cases} \dfrac{y^2 x}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is differentiable at $(0, 0)$. I already know that it's continuous at $(0, 0)$ – I remember that from class.

(a) If we restrict the function $f(x, y)$ to the x-axis, points of the form $(x, 0)$, what does the function look like? Is $f_x(0, 0)$ defined? If so, what does it equal?

(b) Similarly, if we restrict the function $f(x, y)$ to the y-axis, points of the form $(0, y)$, what does the function look like? Is $f_y(0, 0)$ defined? If so, what does it equal?

(c) If f were differentiable at $(0, 0)$, what would its derivative matrix $\mathbf{D}f$ at $(0, 0)$ have to be?

(d) Using that matrix, what would be the instantaneous rate of change of f at $(0, 0)$ going in the direction $\vec{v} = (1, 1)$?

(e) If we restrict the function to the line $y = x$, points of the form (t, t), what does the function look like? How fast is this function changing when $t = 0$?

(f) If f were differentiable, explain how the answers to parts (d) and (e) should be related.

(g) Is f differentiable at $(0, 0)$?

5. Consider the function $f(x, y) = 25 - x^2 - 2y^2$.

(a) Compute $f(2, 3)$, $f_x(2, 3)$ and $f_y(2, 3)$.

The equation of a plane in $\mathbb{R}^3$ is $z = mx + ny + c$, which we could also consider to be the graph of the function $g(x, y) = mx + ny + c$.

(b) Compute $g_x(2, 3)$ and $g_y(2, 3)$.

If we want the plane $z = mx + ny + c$ to approximate the graph of $z = f(x, y)$ above $(2, 3)$ as closely as possible, clearly we'd want:

(i) The plane to pass through the same point as the graph of $f(x, y)$ over $(2, 3)$.

(ii) The plane to have the same instantaneous change in the x-direction as the graph at $(2, 3)$.

(iii) The plane to have the same instantaneous change in the y-direction as the graph at $(2, 3)$.

So,

(c) Find values of m, n, and c so that all these three things are true.

Suppose we pick numbers v_x and v_y and make the vector $\vec{v} = (v_x, v_y)$. The line

$$(2 + t\, v_x,\ 3 + t\, v_y)$$

is a line which passes through $(2, 3)$ when $t = 0$, and heads off in the direction $\vec{v}$.

(d) Compute the function $g(2 + t\, v_x, 3 + t\, v_y)$ of t, and find its derivative when $t = 0$. (Use the values of m, n, and c from part (c).)

(e) Compute the function $f(2 + t\, v_x, 3 + t\, v_y)$ of t, and find its derivative when $t = 0$.

(f) Do the answers to (d) and (e) explain what it means for f to be differentiable at $(2, 3)$? Is f differentiable at $(2, 3)$?

(g) Knowing what it means (from above) for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be differentiable, How would you explain intuitively what it means for a function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ to be differentiable? (SUGGESTION: We can write $\mathbf{F}$ in terms of its three component functions: $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y))$. Maybe differentiablity can be understood in terms of the component functions individually.)

(h) (MINI-BONUS QUESTION) Can you explain why the derivative $\mathbf{DF}$ of a function $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at a point $(x_1, \ldots, x_n)$ should be an $(m \times n)$ matrix?